

# Stochastic Burgers PDEs with random coefficients and a generalization of the Cole-Hopf transformation \*

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## Abstract

Burgers equation is a quasilinear partial differential equation, proposed in 1930's to model the evolution of turbulent fluid motion, which can be *linearized* to the heat equation via the celebrated Cole-Hopf transformation. This paper introduces and studies in detail general versions of stochastic Burgers equation with *random* coefficients, in both forward and backward sense. Concerning the former, the Cole-Hopf transformation still applies and we reduce a forward stochastic Burgers equation to a forward stochastic heat equation that can be treated in a “pathwise” manner. In case of deterministic coefficients, we obtain a *probabilistic* representation of the Cole-Hopf transformation by associating the backward Burgers equation with a system of forward-backward stochastic differential equations. Returning to random coefficients, we exploit this representation in order to establish a stochastic version of the Cole-Hopf transformation. This *generalized* transformation allows us to find solutions to a backward stochastic Burgers equation through a backward stochastic heat equation, subject to additional constraints that reflect the presence of randomness in the coefficients. In both settings, forward and backward, *stochastic* Feynman-Kac formulae are derived for the solutions of the respective stochastic Burgers equations, as well. Finally, applications that illustrate the obtained results are presented to a controllability problem and to a pricing/hedging problem arising from mathematical finance.

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# 1 Introduction

Burgers equation plays a very important rôle in the theory of differential equations and applied mathematics. It is a quasilinear parabolic partial differential equation (PDE) of the form

$$U_t + U U_x = \frac{1}{2} \nu U_{xx}, \quad (1)$$

where  $U(t, x)$  is an unknown velocity field, which is to be determined by the initial condition  $U(0, x) = u_0(x)$ , and  $\nu$  is a viscosity term; hereafter explicit dependence of the subsequent fields is suppressed for ease of notation and subscripts will denote partial derivatives with respect to the corresponding variable. This equation can be considered either in the whole real line ( $x \in \mathbb{R}$ ) or in a bounded interval with Dirichlet or Neumann boundary conditions. It has been proposed in the 1930's by the Dutch scientist J.M. Burgers as a simple model for the dynamics of the Navier-Stokes equations in one spatial dimension [5, 6]. As such it has been used to model the dynamics of one dimensional pressureless turbulence in fluid flows and the hope was that it would help to understand a lot of the intricate structure of this fascinating subject. One of the great breakthroughs in this study was the discovery by J. D. Cole [10] and E. Hopf [26] in the 1950's of a transformation that reduces this equation to the heat equation, thus allowing the derivation of exact solutions in closed form. In particular by the transformation  $U = -\frac{\partial}{\partial x} \ln V$  to a new variable  $V(t, x)$ , where we take without loss of generality that  $\nu = 1$ , the Burgers equation transforms to the linear heat equation

$$V_t = \frac{1}{2} V_{xx}. \quad (2)$$

This transformation, which has been named the Cole-Hopf transformation<sup>1</sup>, allows us to construct explicit analytic solutions for the Burgers equation.

This breakthrough ended temporarily the career of Burgers equation as a modeling tool for turbulence, however, it turned it into a scientific paradigm (a) of a nonlinear PDE that is linearizable with the use of a simple tranformation (in some sense the PDE analogue of the Ricatti equation) and (b) of a benchmark model that can be used to understand the basic features of the interaction between nonlinearity and dissipation. The scientific community soon found many new and interesting applications for the Burgers equation, other than its initial ones in fluid mechanics. For instance, as a model for condensed matter physics, in statistical physics (a continuous time version of ballistic deposition models within the framework of the Kardar-Parisi-Zhang (KPZ) model), in cosmology within what is referred to as the Zel'dovich approximation [43], in the modelling of traffic flow [9], and even in economic theory [24, 25, 42].

However, the Burgers equation developed a parallel, theoretical, and abstract mathematical existence beyond its dominant presence in applications. Motivated by the intention to reinstate Burgers equation as a model for turbulence, the nonlinear dynamics community turned its attention to the *randomly* forced Burgers equation. This introduced a class of quasilinear *stochastic* PDEs

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<sup>1</sup> It is worth noting that a version of Burgers equation was proposed in 1906 by Forsyth, who in fact proposed a version of the linearising transformation, and a similar equation was further proposed by Bateman in 1915. For the history of the Burgers equation and the Cole-Hopf transformation see [19].

(SPDEs) of the form

$$U_t + U U_x = \frac{1}{2} U_{xx} + F(t, x, \omega),$$

where now  $F(\cdot, x, \cdot)$  is a stochastic process that acts as the forcing term. This turns the solution into a *random field*, whose properties are to be determined by the data of the problem, i.e., the forcing term and the initial condition. Depending on the properties of the random forcing term, this equation has to be treated accordingly, i.e., pathwise or in the Itô sense

$$dU = \left( \frac{1}{2} U_{xx} - U U_x \right) dt + Q dW(t),$$

in the case where the forcing term  $F(t, x, \omega)$  can be modelled as an infinite dimensional Brownian motion.

The introduction of *randomness* in Burgers equation produced a number of very interesting new directions; directions connected with dynamical systems aspects of the equation, e.g. existence and properties of invariant measures (see for instance the important contributions by E et al [17] or Goldys and Maslowski [21]), directions related to various questions on the well-posedness of the equation in various functional settings using techniques from infinite dimensional stochastic analysis (see for instance the important contributions of Da Prato, Deboussche, Nualart and others [14, 15, 23, 22, 30, 37]), interesting connections with geometry (see for instance the contributions by Cruzeiro and Malliavin [12] or Davies, Truman, and Zhao [16]), connections with the theory of superprocesses in [3] etc. This theoretical work was inspired by issues related to turbulence (see e.g. [1, 39, 40]) but also led to many new exciting applications (e.g. the work of Kiefer [29] that connects the random Burgers equation with polymer models, see also the review paper of Bec and Khanin [2] on Burgers turbulence and references therein for details on other possible applications).

Another interesting problem related to the Burgers equation, both for the deterministic as well as for the stochastic version, is that of its optimal control or controllability. This is an important problem both from the point of view of theoretical considerations as from the point of view of applications. Temam and coworkers [8] studied this problem from the viewpoint of application in turbulence control and gave some positive answers to the question of whether a feedback control law can be a feasible way to drive the fluid flow to a desired final state. Such questions were further elaborated upon by Da Prato and Debussche [13] with the use of dynamic programming techniques. However, there has been a renewed interest on the issue of controllability for Burgers equation; see for instance [7, 27, 20] for recent related work. These problems bring up the following two interesting directions.

The first one is related to the *well-posedness* of the forward stochastic Burgers equation, complemented with the feedback control law predicted by the dynamic programming equation. Following [13] such a closed loop equation may take the form

$$dU = \left( \frac{1}{2} U_{xx} - U U_x \right) dt + \Phi(t, U)dt + Q dW(t), \quad (3)$$

where  $\Phi(t, U)$  is a functional specifying the feedback control law. The exact form of  $\Phi$  depends on the cost criterion which is to be optimized, and at this point is irrelevant to our discussion.

What is important, though, is that it introduces the need to study more complicated forms of the forward stochastic Burgers equation, that includes a potential like term  $\Phi(t, U)$ . This term may well be a random field itself.

The second one is related to the issue of *controllability*. When one considers the problem of whether it is feasible to find a control procedure  $W(t, U)$  which drives the system to a desired final state  $\xi$  at a given time  $T$ , then it is appropriate to look at the original system as a final value problem with  $U(T, x) = \xi$ , rather than as an initial value problem. For instance, if one wishes to find  $W$  so as to drive the system

$$U_t - U U_x = -\frac{1}{2} U_{xx} + W(t, U)$$

to the desired state  $U(T, x) = \xi$  then one should study the well-posedness of the resulting backward equation in the sense of the existence of a random field  $U$  and a mapping  $W(t, U)$  that satisfy the above problem. There are close connections between the problem of controllability and the well-posedness of this backward problem; see for instance the discussion of the famous Hilbert Uniqueness Method (HUM) proposed by J. L. Lions [32] and J. M. Coron [11] for the control of linear and nonlinear spatially extended systems, respectively. This powerful idea requires a major revision so that it may be applicable to SPDEs, on account of problems related to the adaptivity of the solution to the filtration generated by the noise process when time is reversed... Such questions led to the development of the theory of backward SPDEs (BSPDEs) (see for instance the work by Ma and Yong in [35] and [36] that have recently found applications in [38], [4] or [18] to “pathwise” utility optimization problems emerging from mathematical finance. Using this powerful theory we may consider the problem of controllability of the stochastic Burgers equation, by addressing the problem of well-posedness of a backward stochastic Burgers equation that has the form

$$\begin{aligned} dU &= \left( -\frac{1}{2} U_{xx} - U U_x + \Phi(U, Z) \right) dt + Z dW(t), \\ U(T, x) &= \xi, \end{aligned}$$

where  $\xi$  is a given random final condition, and now we need to look for the pair of random fields  $(U, Z)$ . As expected the unknown random field  $Z$  is related to the control procedure needed to drive the system to the desired final state.

In all the above, the stochasticity was supposed to be an additive term, imposed in the system through the external driving force  $F(t, x, \omega)$ . However, the randomness may well be inserted in the model in terms of *random* coefficients. For instance, one may consider a model which is of the form of the Burgers equation, but with the viscosity term  $\nu$  being a random field rather than a constant parameter. Such a situation may be a good model for fluid turbulence, in which the viscosity term is determined through the distribution of turbulent eddies in the flow. Another case where a Burgers equation with random coefficients could arise, is when we consider a random potential. This may either arise as a physical model (e.g. as a model for a conducting fluid in the presence of external random electromagnetic fields) or in terms of a controlled problem, in the spirit of (3). Other possible motivations could arise in the context of economic theory, and in particular within the context of portfolio selection in market models with random coefficients.

It is therefore the aim of this paper to study *forward* as well as *backward* stochastic Burgers equations with *random* coefficients. Such problems have not been studied extensively in the

literature. One of our main concerns is to study whether a *generalization* of the Cole-Hopf transformation can be obtained, that allows us to transform the *quasilinear* Burgers equation with random coefficients to a *linear* heat equation. If this holds, then we may find solutions via this approach for the Burgers equation possibly in closed form, a fact that will allow us to obtain interesting information on their properties. Such information will provide important input for the relevant models. On the other hand, even if the solution of the heat equation in analytic form is no more possible, due to the presence of the random coefficients, the reduction to a linear equation will allow us to deduce interesting qualitative information on the solutions of the Burger equation, such as comparison principles, Feynman-Kac representations etc.

In the forward case, we employ the Cole-Hopf transformation to a forward version of stochastic Burgers equation with random coefficients, linearizing it to a stochastic heat equation that does not contain a stochastic integral term; that is, it can be solved *pathwise* as a deterministic one. A special version of forward stochastic Burgers equation with constant coefficients was studied in [41]. The backward case is far more delicate and our method of approach is to associate the stochastic Burgers equation with a system of ordinary forward-backward stochastic differential equations (FBSDEs) with random coefficients, through the four step scheme introduced in [33]; an exposition of the theory on FBSDEs may also be found in [35]. To do so, firstly we establish this relation both for the Burgers equation and for its corresponding heat equation in the case of deterministic coefficients, developing a *probabilistic* approach to the Cole-Hopf transformation that allows us to reinterpret it as a *point* transformation between the two associated FBSDE systems. Using then this approach, we *generalize* this transformation to backward stochastic Burgers equations with random coefficients. This generalization is by no means trivial and reflects interesting features concerning the nature of the equation. Through this extended transformation, we find the most general version of backward stochastic Burgers equation that is linearizable and reducible to a stochastic heat equation, subject to additional constraint equations that now appear due to the randomness of the coefficients. Examples of completely solvable backward stochastic Burgers equations are also presented for particular families of random coefficients.

In both cases, forward or backward, the resulting stochastic heat equation is linear and much easier to handle. Thus from the linear system and the generalized Cole-Hopf transformation we construct solutions to stochastic Burgers equations with random coefficients and obtain *stochastic* Feynman-Kac type representations for them. Our results indicate that the backward stochastic Burgers equations, whose study has been neglected so far in the literature, have more intricate structure than their forward counterparts and can lead to interesting applications. In particular, the controllability of a backward stochastic Burgers system that drives it to a predetermined final state amounts to its solvability subject to a suitably selected initial state control. On the other hand, in a tax regulated financial market with a money market and a stock, a small investor may price and hedge a contingent claim, whose future value depends on the volatility of the stock, by selecting his portfolio according to the solution of an FBSDE system and, in turn, to the solution of the associated backward stochastic Burgers equation.

A summary of the paper is as follows. In Section 2 we introduce a forward and a backward general version of stochastic Burgers equation with random coefficients, and present the necessary mathematical preliminaries. Section 3 connects the backward deterministic Burgers equation with a class of FBSDEs, via the four step scheme, developing a probabilistic approach to the celebrated

Cole-Hopf transformation. In Sections 4 and 5 we cope with the linearization of the already introduced forward and backward stochastic Burgers equations with random coefficients, respectively. Stochastic Feynman-Kac formulae are established for their solutions in Section 6. Finally, Section 7 illustrates the obtained results with applications to controllability and mathematical finance.

## 2 Random Burgers Equations: Introduction of Two Models and Mathematical Preliminaries

The main goal of this section is two-fold. The first goal is to introduce two versions of stochastic Burgers equations with *random* coefficients, a forward and a backward one, and motivate them as generalizations of the deterministic Burgers equation in the presence of noise and randomness in the coefficients. The second goal is to set the necessary notation and functional framework, and also present briefly the necessary mathematical tools that will be used in the paper.

### 2.1 Two versions of the random Burgers equation

In contrast with the deterministic case where the forward and the backward problem in time can be related with the use of the simple time inversion transformation  $t \rightarrow -t^2$ , in the stochastic case with or without random coefficients this is no longer true, on account of technical but vital for the nature of the problem difficulties related with the adaptedness of the solution to the filtration generated by the noise process. Therefore, in the stochastic case the forward and the backward problems have to be formulated and treated differently. In both problems, we consider a 1-dimensional Brownian motion  $W(\cdot)$  on some filtered probability space  $(\Omega, \mathcal{F}, P; \mathbb{F})$  with  $\mathbb{F} \triangleq \mathcal{F}(\cdot)$  being the natural filtration generated by  $W(\cdot)$ , augmented by all the  $P$ -null sets in  $\mathcal{F}$ . Generalizations to higher dimensional or even infinite dimensional noise are feasible but are not pursued in the present work. In addition,  $\sigma$ ,  $a$ ,  $g$ ,  $b$ ,  $e$ ,  $s$ ,  $m$ , and  $f$  are assumed to be square integrable, real-valued random fields defined on  $[0, T] \times \mathbb{R} \times \Omega$  for a terminal time  $T > 0$ , such that for fixed  $x \in \mathbb{R}$  they are  $\mathbb{F}$ -progressively measurable.

We consider first the *forward* version of the stochastic Burgers equation with initial condition given by a square integrable random field  $p : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ ; that is,

$$\begin{aligned} dU = & \left[ \frac{1}{2} \sigma^2(t, x) U_{xx} + a(t, x) U U_x + g(t, x) U \Psi(t, U) + b(t, x) U_x \right. \\ & \left. + e(t, x) U + s(t, x) \Psi_x(t, U) + m(t, x) \Psi(t, U) + f(t, x) \right] dt \\ & + \Psi(t, U) dW(t), \quad 0 < t \leq T, \quad x \in \mathbb{R}, \\ U(0, x) = & p(x), \quad x \in \mathbb{R}. \end{aligned} \tag{4}$$

Here  $\Psi(t, U)$  is a *known* function (or even random field) of  $U$ , pre-described by the form of the model. It reflects the effect of the fluctuations on the evolution of  $U$ , i.e., how the effect of  $W(\cdot)$

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<sup>2</sup>eventhough the parabolic nature of the equation may introduce certain difficulties related to the unique continuation of the solution for negative times

transfers to an effect on  $U$ . In the simple case where  $\Psi(t, U)$  is a constant, we obtain a stochastic Burgers equation with additive noise.

The motivation for equation (4) is rather straightforward. One may consider any physical situation where the evolution of the state of the system in question is modeled in terms of Burgers equation. Then assume that this system is subject to external sources, whose behavior is random and subject to fluctuations (this accounts for the Wiener process term in the above system), while at the same time the coefficients of the model are also subject to uncertainties. This uncertainty is introduced by the random coefficients, in a sense that the coefficients of the model are functions, rather than constants, of a random process that evolves in parallel with the state of the system. This random process depends on the external noise term, therefore we assume that the random coefficients are functions which are adapted to the filtration generated by the Wiener process.

To illustrate the above points let us consider the example of turbulence modeling: the Wiener process models a body force for the fluid which presents fluctuations around an average body force. The random coefficients would correspond to a randomly changing viscosity term in the fluid. This is a very reasonable modeling assumption since for example the effective viscosity of the fluid depends on the eddies formed in the turbulent flow which in turn depend on the random fluid velocity, which in turns depends on the external random body force. The above discussion can be easily transferred to other models. In conclusion we ought to comment on the formulation of a forward equation. We assume that the state of the system is known at  $t = 0$ , and our aim through this model is to predict the possible future states of the system and provide information on their statistical properties. This problem will be a useful model for a number of physical situations where the Burgers equation arises.

We now introduce the *backward* problem. Here we assume that we know the final state of the system, which is the square integrable  $\mathcal{F}_T$ -measurable random field  $p : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ , and we wish to find which initial condition must be chosen in order to drive the system at time  $T$  to this state. The proper formulation of this problem is in terms of the BSPDE:

$$\begin{aligned} dU = & \left[ -\frac{1}{2} \sigma^2(t, x) U_{xx} + a(t, x) U U_x + g(t, x) U \Psi^U + b(t, x) U_x \right. \\ & \left. + e(t, x) U + s(t, x) \Psi_x^U + m(t, x) \Psi^U + f(t, x) \right] dt \\ & + \Psi^U dW(t), \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\ U(T, x) = & p(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5}$$

It should be stressed here that now the unknowns are the pair of random fields  $(U, \Psi^U)$  and not just  $U$ . Therefore in contrast to what holds for the forward problem (4),  $\Psi(t, U)$  can no longer be thought of as given data to the problem, i.e. as pre-described by the model, but now it has to be specified as part of the solution of the problem. What is even more interesting is that  $\Psi^U$  is *uniquely* determined by  $U$ , as a properly defined functional of  $U$  and not necessarily as a function of  $U$ . In some sense  $\Psi^U$  has to be interpreted as the (unique) auxiliary process needed to drive the system to the desired random final state. Therefore, it may have the interpretation of a *control procedure* which acts on the system so as to drive it to a desired state. Furthermore,

the compensation of this auxiliary process, allows us to obtain a solution to the backward problem which is adapted to the filtration generated by the Wiener process.

The important motivation for this problem arises from its connection with control problems. By the discussion above, one observes immediately that the backward problem is intimately related to the problem of controllability of the stochastic Burgers equation. This problem can be stated as follows: given an initial state can we drive the system by the proper control procedure to a desired final state which is random but with prescribed statistical properties? As we have stated the problem, it allows us to characterize this initial state that will drive us to the desired final state, and at the same time through  $\Psi^U$  characterize the control procedure needed in the accomplishment of this task. The connection is much deeper; in particular, through the generalization of the Pontryagin maximum principle in the context of SPDEs one may show that the dual (adjoint) system associated with a large variety of optimal control problems will have a form closely related to (5).

## 2.2 Notation, functional setting, and some important preliminary results

Throughout this paper, denote by  $C^k(\mathbb{R}^n)$  for any integer  $k \geq 0$  the set of real-valued functions on  $\mathbb{R}^n$  that are continuously differentiable up to order  $k$ . In addition, for any  $1 \leq p \leq \infty$ , any Banach space  $\mathbb{X}$  with norm  $\|\cdot\|_{\mathbb{X}}$ , and any sub- $\sigma$ -algebra  $\mathcal{G} \subseteq \mathcal{F}$ , denote by

- $\mathbb{L}_{\mathcal{G}}^p(\Omega, \mathbb{X})$  the set of all  $\mathbb{X}$ -valued,  $\mathcal{G}$ -measurable random variables  $X$  such that  $E\|X\|_{\mathbb{X}}^p < \infty$  ;
- $\mathbb{L}_{\mathbb{F}}^p(0, T; \mathbb{X})$  the set of all  $\mathbb{F}$ -progressively measurable,  $\mathbb{X}$ -valued processes  $X : [0, T] \times \Omega \rightarrow \mathbb{X}$  such that  $\int_0^T \|X(t)\|_{\mathbb{X}}^p dt < \infty$ , a.s.;
- $\mathbb{L}_{\mathbb{F}}^p(0, T; \mathbb{L}^p(\Omega; \mathbb{X}))$  the set of all  $\mathbb{F}$ -progressively measurable,  $\mathbb{X}$ -valued processes  $X : [0, T] \times \Omega \rightarrow \mathbb{X}$  such that  $\int_0^T E\|X(t)\|_{\mathbb{X}}^p dt < \infty$  ;
- $C_{\mathbb{F}}([0, T]; \mathbb{X})$  the set of all continuous,  $\mathbb{F}$ -adapted processes  $X(\cdot, \omega) : [0, T] \rightarrow \mathbb{X}$  for  $P$ -a.e.  $\omega \in \Omega$ .

Moreover, define similarly the set  $C_{\mathbb{F}}([0, T]; \mathbb{L}^p(\Omega; \mathbb{X}))$ , and let  $\mathbb{R}^+$  stand for the positive real numbers. Finally, for every random field  $F : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$  denote by  $A^F, \Psi^F : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$  the random fields for which  $F$  obtains the semimartingale decomposition

$$F(t, x) = F(0, x) + \int_0^t A^F(s, x) ds + \int_0^t \Psi^F(s, x) dW(s) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n,$$

almost surely. Assume a similar notation for stochastic processes as well..

In order to carry out computations regarding the change-of-variable formula for random fields, we recall the following useful implication of the generalized Itô-Kunita-Wentzell (IKW) formula (e.g. [31], Section 3.3, pp 92-93).



**Proposition 2.1.** *Suppose that the random field  $F : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$  is of class  $C^{0,2}([0, T] \times \mathbb{R}^n)$  and  $A^F, \Psi^F : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$  are  $\mathbb{F}$ -adapted random fields of class  $C^{0,1}([0, T] \times \mathbb{R}^n)$  and  $C^{0,2}([0, T] \times \mathbb{R}^n)$ , respectively. Furthermore, let  $X = (X^{(1)}, \dots, X^{(n)})^*$  be a vector of continuous semimartingales, where  $A^{X^{(i)}}(\cdot)$  is an almost surely integrable process and  $\Psi^{X^{(i)}}(\cdot)$  is an  $\mathbb{F}$ -progressively measurable, almost surely square integrable process. Then  $F(\cdot, X(\cdot))$  is also a continuous semimartingale, with decomposition*

$$\begin{aligned}
F(t, X(t)) &= F(0, X(0)) + \int_0^t A^F(s, X(s)) ds + \int_0^t \Psi^F(s, X(s)) dW(s) \\
&+ \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} F(s, X(s)) A^{X^{(i)}}(s) ds \\
&+ \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} F(s, X(s)) \Psi^{X^{(i)}}(s) dW(s) \\
&+ \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} \Psi^F(s, X(s)) \Psi^{X^{(i)}}(s) ds \\
&+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} F(s, X(s)) \Psi^{X^{(i)}}(s) \Psi^{X^{(j)}}(s) ds
\end{aligned} \tag{6}$$

for every  $0 \leq t \leq T$ .

### 3 Connection of Backward Burgers Equation with FBSDEs and a Probabilistic Approach to the Cole-Hopf Transformation

For a given function  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $\sigma \neq 0$ , consider the backward *deterministic* Burger's equation of the form

$$\begin{aligned}
U_t + \frac{1}{2} \sigma^2 U_{xx} - \sigma^2 U U_x &= 0, \quad 0 < t \leq T, \quad x \in \mathbb{R}, \\
U(T, x) &= p(x), \quad x \in \mathbb{R}.
\end{aligned} \tag{7}$$

It is well known that using a transformation to a new variable  $V : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $U = -\frac{\partial}{\partial x} \ln V$ , equation (7) is linearized assuming the form of the heat equation

$$\begin{aligned}
V_t + \frac{1}{2} \sigma^2 V_{xx} &= 0, \quad 0 < t \leq T, \quad x \in \mathbb{R}, \\
V(T, x) &= e^{-\int p(x) dx} \triangleq q(x), \quad x \in \mathbb{R}.
\end{aligned} \tag{8}$$

This is the celebrated Cole-Hopf transformation, through which one may construct solutions of Burgers equation using appropriate solutions of the heat equation.

Let us revisit this linearization from a probabilistic viewpoint. For any  $x \in \mathbb{R}$ , consider the system of FBSDEs

$$\begin{aligned} dX(t) &= \sigma dW(t), \quad 0 < t \leq T, \\ dY(t) &= \sigma Y(t) Z(t) dt + Z(t) dW(t), \quad 0 \leq t < T, \\ X(0) &= x, \quad Y(T) = p(X(T)), \end{aligned} \tag{9}$$

where  $X(\cdot)$  is the forward process,  $Y(\cdot)$  is the backward process, and  $Z(\cdot)$  is the auxiliary process needed for the well-posedness of the problem, each defined on  $[0, T] \times \Omega$ . If one looks for a Markovian solution to this problem of the form  $Y(t) = U(t, X(t))$  then a simple application of Itô's formula yields both that the deterministic function  $U$  should satisfy equation (7) and the relationship  $Z(t) = \sigma U_x(t, X(t))$ . Consider further the FBSDE

$$\begin{aligned} dx(t) &= \sigma dW(t), \quad 0 < t \leq T, \\ dy(t) &= z(t) dW(t), \quad 0 \leq t < T, \\ x(0) &= x, \quad y(T) = q(x(T)) \end{aligned} \tag{10}$$

for the  $\mathbb{F}$ -adapted processes  $x, y, z : [0, T] \times \Omega \rightarrow \mathbb{R}$ , and follow the same steps as above to verify that for a Markovian solution of the form  $y(t) = V(t, x(t))$  the deterministic function  $V$  should satisfy equation (8) and  $z(t) = \sigma V_x(t, x(t))$ .

The following issue is addressed. Taking into account that the PDEs (7) and (8) are related via the Cole-Hopf transformation, we should expect that the probabilistic systems (9) and (10), giving rise to these equations respectively, must be related as well. Conversely, if we find a point transformation between the variables  $(X, Y, Z)$  and  $(x, y, z)$  then we could use it to reproduce the Cole-Hopf transformation that allows us to get from the Burgers equation to the heat equation. In particular, we consider the point transformation between the solutions of (9) and (10):

$$\begin{aligned} X &= x, \\ Y &= \mathcal{Y}(x, y, z), \\ Z &= \mathcal{Z}(x, y, z), \end{aligned} \tag{11}$$

where  $\mathcal{Y}, \mathcal{Z} : \mathbb{R}^3 \rightarrow \mathbb{R}$  are deterministic functions to be determined. Then we have the following result.

**Proposition 3.1.** *Suppose that the process triplet  $(x, y, z)$  solves (10), and the function  $\mathcal{Y}$  belongs to  $\mathcal{C}^{2,2,2}(\mathbb{R}^3)$  and satisfies the PDE*

$$\begin{aligned} \frac{1}{2} \sigma^2 \mathcal{Y}_{xx} + \frac{1}{2} z^2 \mathcal{Y}_{yy} + \frac{1}{2} h^2 \mathcal{Y}_{zz} + \sigma z \mathcal{Y}_{xy} + h z \mathcal{Y}_{yz} + \sigma h \mathcal{Y}_{xz} \\ - \sigma^2 \mathcal{Y} \mathcal{Y}_x - \sigma z \mathcal{Y} \mathcal{Y}_y - \sigma h \mathcal{Y} \mathcal{Y}_z = 0 \end{aligned} \tag{12}$$

for every  $h \in \mathbb{R}$ . Furthermore let

$$\begin{aligned} \mathcal{Z}(x(t), y(t), z(t)) &= \sigma \mathcal{Y}_x(x(t), y(t), z(t)) + z(t) \mathcal{Y}_y(x(t), y(t), z(t)) \\ &\quad + h(t) \mathcal{Y}_z(x(t), y(t), z(t)), \end{aligned} \tag{13}$$

in terms of the process  $h : [0, T] \times \Omega \rightarrow \mathbb{R}$  such that  $z(\cdot)$  has the local martingale representation  $dz(t) = h(t) dW(t)$ ; cf. Remark 3.2. Then the process triplet  $(X, Y, Z)$  defined through the point transformation of (11) solves (9).

*Proof.* Take the Itô differential of  $Y(t) = \mathcal{Y}(x(t), y(t), z(t))$  to get

$$dY(t) = I(x(t), y(t), z(t); h(t)) dt + K(x(t), y(t), z(t); h(t)) dW(t), \quad (14)$$

where

$$\begin{aligned} I(x, y, z; h) &\triangleq \frac{1}{2} \sigma^2 \mathcal{Y}_{xx}(x, y, z) + \frac{1}{2} z^2 \mathcal{Y}_{yy}(x, y, z) + \frac{1}{2} h^2 \mathcal{Y}_{zz}(x, y, z) \\ &\quad + \sigma z \mathcal{Y}_{xy}(x, y, z) + h z \mathcal{Y}_{zy}(x, y, z) + \sigma h \mathcal{Y}_{xz}(x, y, z), \\ K(x, y, z; h) &\triangleq \sigma \mathcal{Y}_x(x, y, z) + z \mathcal{Y}_y(x, y, z) + h \mathcal{Y}_z(x, y, z). \end{aligned}$$

Thanks to (12) and (13), these functions satisfy the relationships

$$\begin{aligned} I(x(t), y(t), z(t); h(t)) &= \sigma \mathcal{Y}(x(t), y(t), z(t)) \mathcal{Z}(x(t), y(t), z(t)), \\ K(x(t), y(t), z(t); h(t)) &= \mathcal{Z}(x(t), y(t), z(t)), \end{aligned}$$

and substituting them to (14) we obtain the stated result.  $\square$

**Remark 3.2.** Note that applying Itô's differential to  $z(t) = \sigma V_x(t, x(t))$ , in conjunction with (8), allows us to take  $dz(t) = h(t) dW(t)$  in the previous proposition.

**Remark 3.3.** One can easily verify that a solution of equation (12) for any  $h \in \mathbb{R}$  is the rational function  $\mathcal{Y}(x, y, z) = -z/\sigma y$ . This interpreted in terms of the solutions  $U$  and  $V$  of (7) and (8), respectively, corresponds to the Cole-Hopf transformation, since

$$U(t, x(t)) = Y(t) = \mathcal{Y}(x(t), y(t), z(t)) = -\frac{z(t)}{\sigma y(t)} = -\frac{V_x(t, x(t))}{V(t, x(t))}.$$

Hence, making use of probabilistic tools we managed to derive an alternative representation of this transformation in terms of a solution of the PDE (12).

## 4 Linearization of the Forward Random Burgers Equation

In this section we consider the *forward* version of the stochastic Burgers equation (4) using the notation  $\Psi(t, U) = \Psi^U$  and stress the fact that here  $\Psi^U$  is a pre-described function of  $U$ . It is the object of the subsequent result to find the general form of the random coefficients of the above equation as well as of the function  $\Psi^U$  such that this forward SPDE (FSPDE) can be transformed to a linear heat equation through the use of the Cole-Hopf transformation. Therefore, if we obtain somehow positive solutions for the linear one we may obtain solutions for the original Burgers equation with random coefficients by means of the Cole-Hopf transformation.

**Theorem 4.1.** *The FSPDE*

$$\begin{aligned}
dU = & \left[ \frac{1}{2} \sigma^2(t, x) U_{xx} + a(t, x) U U_x + g(t, x) U \Psi^U + b(t, x) U_x + e(t, x) U \right. \\
& \left. + s(t, x) \Psi_x^U + m(t, x) \Psi^U + f(t, x) \right] dt \\
& + \Psi^U dW(t), \quad 0 < t \leq T, \quad x \in \mathbb{R}, \\
U(0, x) = & p(x), \quad x \in \mathbb{R}
\end{aligned} \tag{15}$$

is linearizable through the Cole-Hopf transformation to a stochastic heat equation if  $\sigma, a, g$ , and  $s$  are stochastic processes of the form

$$a(\cdot) \triangleq \sigma^2(\cdot), \quad g(\cdot) \triangleq \sigma(\cdot), \quad \text{and} \quad s(\cdot) \triangleq -\sigma(\cdot), \tag{16}$$

the function  $\Psi^U$  is given by

$$\Psi^U(t, x) \triangleq \ell(t) U_x(t, x) \quad \forall (t, x) \in (0, T] \times \mathbb{R} \tag{17}$$

for a stochastic process  $\ell : [0, T] \times \Omega \rightarrow \mathbb{R}$  defined by

$$\ell(\cdot) \triangleq -2 \sigma(\cdot), \tag{18}$$

and finally

$$e(t, x) \triangleq b_x(t, x) + \ell(t) m(t, x) \quad \forall (t, x) \in [0, T] \times \mathbb{R}; \tag{19}$$

in particular, if  $U \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^2(\mathbb{R})))$  is a random field that solves FSPDE

$$\begin{aligned}
dU = & \left[ \frac{5}{2} \sigma^2(t) U_{xx} - \sigma^2(t) U U_x + \left( b(t, x) - 2\sigma(t) m(t, x) \right) U_x \right. \\
& \left. + e(t, x) U + f(t, x) \right] dt + \ell(t) U_x dW(t), \quad 0 < t \leq T, \quad x \in \mathbb{R}, \\
U(0, x) = & p(x), \quad x \in \mathbb{R},
\end{aligned} \tag{20}$$

then the random field  $V$  which is given by the Cole-Hopf transformation is of class

$C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}; \mathbb{R}^+)))$  and solves FSPDE

$$\begin{aligned} dV = & \left\{ \frac{5}{2} \sigma^2(t) V_{xx} + k(t, x) V_x - c(t, x) V \right\} dt \\ & + \ell(t) V_x dW(t), \quad 0 < t \leq T, \quad x \in \mathbb{R}, \end{aligned} \quad (21)$$

$$V(0, x) = e^{-\int p(x) dx} \triangleq q(x), \quad x \in \mathbb{R},$$

where

$$c(t, x) \triangleq \int f(t, x) dx + \bar{c}(t) \quad \text{and} \quad k(t, x) \triangleq b(t, x) + \ell(t) m(t, x) \quad (22)$$

for every  $(t, x) \in [0, T] \times \mathbb{R}$  and some process  $\bar{c}(\cdot)$ . Furthermore, the latter admits a solution of the form

$$V(t, x) \triangleq G(t, x + H(t)), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}, \quad (23)$$

where

$$H(t) \triangleq \int_0^t \ell(s) dW(s), \quad 0 \leq t \leq T, \quad (24)$$

in terms of a positive random field  $G$  of class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}; \mathbb{R}^+)))$  that solves FSPDE

$$\begin{aligned} dG = & \left\{ \frac{1}{2} \sigma^2(t) G_{xx}(t, x) + k(t, x - H(t)) G_x(t, x) \right. \\ & \left. - c(t, x - H(t)) G(t, x) \right\} dt \quad \text{on } (0, T] \times \mathbb{R}, \end{aligned} \quad (25)$$

$$G(0, x) = q(x) \quad \text{on } \mathbb{R}.$$

*Proof.* We first establish the definitions of (16)-(19), according to which the FSPDE (20) transforms to the FSPDE (21). To this end, consider the Cole-Hopf transformation  $U = -\frac{\partial}{\partial x} \ln V$ , in terms of a *strictly positive* random field  $V \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}; \mathbb{R}^+)))$  with a semimartingale decomposition such that  $\Psi^V \in \mathbb{L}_{\mathbb{F}}^2(0, T; C^3(\mathbb{R}))$ . For every  $0 < t \leq T$ , Itô's formula and the uniqueness of the semimartingale decomposition imply that

$$dU = -\frac{\partial}{\partial x} \left[ \frac{dV}{V} - \frac{(\Psi^V)^2}{2V^2} dt \right] \quad \text{and} \quad \Psi^U = -\frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right), \quad (26)$$

and substituting back into (15) we have

$$\begin{aligned}
-\frac{\partial}{\partial x} \left[ \frac{dV}{V} - \frac{(\Psi^V)^2}{2V^2} dt \right] = & \left\{ -\frac{1}{2} \sigma^2(t, x) \frac{\partial^3}{\partial x^3} \ln V + a(t, x) \frac{\partial}{\partial x} \ln V \frac{\partial^2}{\partial x^2} \ln V \right. \\
& + g(t, x) \frac{\partial}{\partial x} \ln V \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) - b(t, x) \frac{\partial^2}{\partial x^2} \ln V \\
& - e(t, x) \frac{\partial}{\partial x} \ln V - s(t, x) \frac{\partial^2}{\partial x^2} \left( \frac{\Psi^V}{V} \right) \\
& \left. - m(t, x) \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) + f(t, x) \right\} dt \\
& - \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) dW(t),
\end{aligned} \tag{27}$$

subject to the initial condition of (21). Inspired by the methodology implied by the Cole-Hopf transformation in the linearization of (1), we accordingly integrate the last equation with respect to  $x$  and assume that  $\sigma, a, g$ , and  $s$  are independent of the spatial variable, i.e. are stochastic processes, to arrive at

$$\begin{aligned}
dV = & \left\{ \frac{1}{2} \sigma^2(t) V_{xx} - \frac{1}{2} (\sigma(t)^2 + a(t)) \frac{V_x^2}{V} - g(t) V \int \frac{V_x}{V} \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) dx \right. \\
& + V \left[ \int b(t, x) \frac{\partial^2}{\partial x^2} \ln V dx + \int e(t, x) \frac{V_x}{V} dx \right. \\
& \left. \left. + \int m(t, x) \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) dx \right] \right. \\
& \left. + s(t) \Psi_x^V - s(t) \frac{\Psi^V V_x}{V} - V \int f(t, x) dx + \frac{(\Psi^V)^2}{2V} \right\} dt + \Psi^V dW(t).
\end{aligned} \tag{28}$$

This is an FSPDE for the random field  $V$  where  $\Psi^V$  is a function of  $V$ , the exact form of which depends on the choice of the function  $\Psi^U$  through the second equation of (26). This is in general a nonlinear FSPDE, whose form can be as complicated or even worse than the original Burgers equation. However, to end up with a linear FSPDE (by analogy to what happened in the deterministic case where it reduced to the heat equation (2)), it suffices to consider the special ansatz of (17) and (18) for the function  $\Psi^U$ , which also gives that

$$\Psi^V(t, x) = \ell(t) V_x(t, x) \quad \forall (t, x) \in (0, T] \times \mathbb{R}. \tag{29}$$

We show now that this choice of the function  $\Psi^U$  (leading to the equivalent choice (29) of the function  $\Psi^V$ ) is enough to turn FSPDE (28) for the evolution of  $V$  into a linear one of the heat type. Indeed, employ equation (29) to reformulate the integral expressions of (28) as

$$\int \frac{V_x}{V} \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) dx = \frac{\ell(t)}{2} \left( \frac{V_x}{V} \right)^2$$

and, in conjunction with integration-by-parts, as

$$\begin{aligned} & \int b(t, x) \frac{\partial^2}{\partial x^2} \ln V dx + \int e(t, x) \frac{V_x}{V} dx + \int m(t, x) \frac{\partial}{\partial x} \left( \frac{\Psi^V}{V} \right) dx \\ &= \left[ b(t, x) + \ell(t) m(t, x) \right] \frac{V_x}{V} + \int \left[ e(t, x) - b_x(t, x) - \ell(t) m_x(t, x) \right] \frac{V_x}{V} dx. \end{aligned}$$

Hence, FSPDE (28) becomes

$$\begin{aligned} dV = & \left\{ \frac{1}{2} \left( \sigma^2(t) + 2s(t)\ell(t) \right) V_{xx} + \left[ b(t, x) + \ell(t) m(t, x) \right] V_x \right. \\ & - \frac{1}{2} \left[ \sigma^2(t) + a(t) + (g(t) + 2s(t))\ell(t) - \ell^2(t) \right] \frac{V^2}{V} \\ & + V \int \left[ e(t, x) - b_x(t, x) - \ell(t) m_x(t, x) \right] \frac{V_x}{V} dx \\ & \left. - V \int f(t, x) dx \right\} dt + \ell(t) V_x dW(t). \end{aligned}$$

In order for this equation to be linear with respect to the unknown field  $V$ , the coefficients in front of the terms  $V_x^2/V$  and  $V_x/V$  should vanish. Therefore, we consider that the stochastic process coefficients of FSPDE (15) are expressed in terms of  $\sigma(\cdot)$  and, in particular, are defined as in (16). Regarding the coefficient of the term  $V_x^2/V$ , this is the case by assuming either  $\ell(\cdot) = \sigma(\cdot)$  or  $\ell(\cdot) = -2\sigma(\cdot)$ . We reject the first choice, which leads to definition (18), since it reduces the last FSPDE into an ill-posed parabolic problem; specifically, for this choice the parabolic term comes with a negative diffusion coefficient. We may further assume the definition of (19) to eliminate the coefficient of the term  $V_x/V$  and finally reach the forward linear stochastic heat type equation of (21), given the definitions of (22).

To prove the second assertion of the theorem, we seek for positive solutions of (21) that have the form of (23). Here, we assume that  $G$  is a random field of class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}; \mathbb{R}^+)))$  with a semimartingale decomposition in terms of the pair  $(A^G, \Psi^G) \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^1(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$ . Employing the generalized IKW formula to (23), in conjunction with (24), we obtain

$$\begin{aligned} dV = & \left\{ A^G(t, x + H(t)) + \frac{1}{2} G_{xx}(t, x + H(t)) (\Psi^H(t))^2 \right. \\ & \left. + \Psi^H(t) \Psi_x^G(t, x + H(t)) \right\} dt \\ & + \left\{ \Psi^H(t) G_x(t, x + H(t)) + \Psi^G(t, x + H(t)) \right\} dW(t) \end{aligned}$$

for every  $(t, x) \in (0, T] \times \mathbb{R}$ . Making also use of (23), we rewrite (21) as

$$dV = \left\{ \frac{5}{2} \sigma^2(t) G_{xx}(t, x + H(t)) + k(t, x) G_x(t, x + H(t)) \right. \\ \left. - c(t, x) G(t, x + H(t)) \right\} dt + \ell(t) G_x(t, x + H(t)) dW(t),$$

for every  $(t, x) \in (0, T] \times \mathbb{R}$ . A comparison between the last two equations leads to the definitions

$$A^G(t, x) \triangleq \frac{1}{2} \sigma^2(t) G_{xx}(t, x) + k(t, x - H(t)) G_x(t, x) - c(t, x - H(t)) G(t, x), \\ \Psi^G(\cdot, \cdot) \triangleq 0,$$

where (18) has been used as well. Finally, substituting them in the semimartingale decomposition of  $G$ , we derive the FSPDE of (25); the initial condition comes from (21) and (23) for  $t = 0$ . Regarding the positivity of  $G$  see the following remark.  $\square$

**Remark 4.2.** *FSPDE (25) is a linear PDE with random coefficients, but does not contain any stochastic integral. This means that it can be treated pathwise rather than in an Itô integration sense, a fact that simplifies its analysis immensely, and allows us to use qualitative results, e.g. maximum principles, to study properties of its solutions like positivity, monotonicity etc. In fact, for the special case of constant coefficients with  $k(\cdot, \cdot) = c(\cdot, \cdot) = 0$  this PDE reduces to (2).*

## 5 Linearization of the Backward Random Burgers Equation

In Section 3 we showed how the backward deterministic Burgers equation can be linearized using its connection with a system of FBSDEs, through which we obtained an alternative probabilistic derivation of the celebrated Cole-Hopf transformation. In the present section we address the question of whether this analysis can be generalized for the *backward* stochastic Burgers equation with random coefficients of (5). It will turn out that in order to answer this problem it is no longer enough to use the Cole-Hopf transformation that works for the deterministic or the forward random problem but rather we have to extend it in such a fashion as to take into account the intricate nature of the backward problem, and the presence of the related unknown process pair  $(U, \Psi^U)$ .

For the simplicity of our mathematical program which aims to the linearization of (5), we also assume that the coefficients are bounded and infinitely differentiable with respect to the spatial variable  $x$ , with all their partial derivatives bounded as well. Carrying out though our analysis, we shall see that both weaker regularity conditions in  $x$  may be imposed and explicit relationships among the coefficients will be in need.

### 5.1 Connection with FBSDEs of random coefficients

We consider the forward process  $X(\cdot)$  that satisfies  $X(t) = x + \int_0^t \sigma(s, X(s)) dW(s)$  for  $(t, x) \in [0, T] \times \mathbb{R}$ , and seek for the corresponding backward given by  $Y(t) = U(t, X(t))$  in terms of a



solution  $(U, \Psi^U) \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$  of BSPDE (5). Note here that we have considered an additional order of smoothness for the pair  $(U, \Psi^U)$  than is required to constitute a solution of (5). Nevertheless, this extra assumption allows the application of the generalized IKW formula, which coupled with (5) yields

$$\begin{aligned} dY(t) = & \left\{ \left[ a(t, X(t))U_x(t, X(t)) + g(t, X(t))\Psi^U(t, X(t)) \right] U(t, X(t)) \right. \\ & + b(t, X(t))U_x(t, X(t)) + e(t, X(t))U(t, X(t)) \\ & + \left( s(t, X(t)) + \sigma(t, X(t)) \right) \Psi_x^U(t, X(t)) + m(t, X(t))\Psi^U(t, X(t)) \\ & \left. + f(t, X(t)) \right\} dt + \left[ \sigma(t, X(t))U_x(t, X(t)) + \Psi^U(t, X(t)) \right] dW(t), \end{aligned}$$

for  $0 \leq t < T$ . In order both to be consistent with the form of the deterministic Burgers equation of (7) and to avoid matters of technical fuss in our following analysis, we shall consider that  $g(\cdot) = \sigma(\cdot) \neq 0$  is a stochastic process. Additionally, by setting

$$a(t, x) \triangleq g(t)\sigma(t), \quad b(t, x) \triangleq \sigma(t)m(t, x), \quad \text{and} \quad s(t, x) \triangleq -\sigma(t) \quad (30)$$

for  $(t, x) \in [0, T] \times \mathbb{R}$ , we obtain the first part of the following result.

**Proposition 5.1.** *(i) Suppose that the pair of random fields  $(U, \Psi^U)$  is of class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$  and solves the stochastic Burgers type BSPDE*

$$\begin{aligned} dU = & \left[ -\frac{1}{2} \sigma^2(t)U_{xx} + \sigma^2(t)UU_x + \sigma(t)U\Psi^U + \sigma(t)m(t, x)U_x \right. \\ & \left. + e(t, x)U - \sigma(t)\Psi_x^U + m(t, x)\Psi^U + f(t, x) \right] dt \\ & + \Psi^U dW(t), \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\ U(T, x) = & p(x), \quad x \in \mathbb{R}. \end{aligned} \quad (31)$$

Then the triplet  $(X, Y, Z)$  of stochastic processes, given by

$$\begin{aligned} X(t) & \triangleq x + \int_0^t \sigma(s) dW(s), \quad Y(t) \triangleq U(t, X(t)), \\ \text{and} \quad Z(t) & \triangleq \sigma(t)U_x(t, X(t)) + \Psi^U(t, X(t)) \end{aligned} \quad (32)$$

for every  $t \in [0, T]$ , satisfies the FBSDEs

$$\begin{aligned} dX(t) & = \sigma(t)dW(t), \quad 0 < t \leq T, \\ dY(t) & = \left[ \sigma(t)Y(t)Z(t) + e(t, X(t))Y(t) + m(t, X(t))Z(t) \right. \\ & \quad \left. + f(t, X(t)) \right] dt + Z(t)dW(t), \quad 0 \leq t < T, \\ X(0) & = x, \quad Y(T) = p(X(T)). \end{aligned} \quad (33)$$

(ii) Consider that the pair of random fields  $(V, \Psi^V)$  belongs to the class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^3(\mathbb{R}))$  and satisfies the stochastic heat type BSPDE

$$\begin{aligned} dV &= \left\{ -\frac{1}{2} \sigma^2(t) V_{xx} + \sigma(t) d(t, x) V_x - \sigma(t) \Psi_x^V + d(t, x) \Psi^V \right. \\ &\quad \left. + c(t, x) V \right\} dt + \Psi^V dW(t), \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\ V(T, x) &= q(x), \quad x \in \mathbb{R}. \end{aligned} \quad (34)$$

Then the triplet  $(x, y, z)$  of stochastic processes, defined by

$$\begin{aligned} x(t) &\triangleq x + \int_0^t \sigma(s) dW(s), \quad y(t) \triangleq V(t, x(t)), \\ \text{and} \quad z(t) &\triangleq \sigma(t) V_x(t, x(t)) + \Psi^V(t, x(t)) \end{aligned} \quad (35)$$

for every  $t \in [0, T]$ , solves the FBSDEs

$$\begin{aligned} dx(t) &= \sigma(t) dW(t), \quad 0 < t \leq T, \\ dy(t) &= \left[ c(t, x(t)) y(t) + d(t, x(t)) z(t) \right] dt + z(t) dW(t), \quad 0 \leq t < T, \\ x(0) &= x, \quad y(T) = q(x(T)). \end{aligned} \quad (36)$$

*Proof.* We need only to show (ii). Indeed, the BSPDE (36) follows readily through (34), (35), and the generalized IKW formula.  $\square$

The first part of this proposition provides a solution to the FBSDEs (33) by means of a solution of the Burgers BSPDE (31). In Section 3 though, we saw that the Cole-Hopf transformation reduces the Burgers equation (7) to the linear heat equation (8). Therefore, its next statement addresses the most general type of a backward linear stochastic heat equation (cf. (34)) that is associated with an FBSDE system (cf. (36)).

## 5.2 Characterization of the generalized Cole-Hopf transformation in terms of an SPDE

Our goal in this subsection is to investigate whether there exists a *stochastic* version of the Cole-Hopf that reduces (31) to (34) for appropriate random fields  $c, d : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ . According to Remark 3.3, in the deterministic case the Cole-Hopf transformation is represented by the point transformation (11). Thus, by analogy we shall seek for a point transformation of the form

$$\begin{aligned} X(t) &= x(t), \\ Y(t) &= \mathcal{Y}(t, x(t), y(t), z(t)), \\ Z(t) &= \mathcal{Z}(t, x(t), y(t), z(t)), \end{aligned} \quad (37)$$

that relates the solutions  $(X, Y, Z)$  and  $(x, y, z)$  of (33) and (36) respectively, in terms of suitable random fields  $\mathcal{Y}, \mathcal{Z} : [0, T] \times \mathbb{R}^3 \times \Omega \rightarrow \mathbb{R}$ . This transformation will provide explicitly the Cole-Hopf one in the generalized case of *stochastic* coefficients.

**Assumption 5.2.** *Let the stochastic coefficient  $\sigma(\cdot)$  obtain a semimartingale decomposition, where  $A^\sigma(\cdot)$  is an integrable a.s.,  $\mathbb{F}$ -progressively measurable stochastic process, and  $\Psi^\sigma(\cdot)$  is a square-integrable a.s.,  $\mathbb{F}$ -progressively measurable one. Furthermore, let the latter process obtain a similar semimartingale decomposition in terms of the stochastic processes  $A^{\Psi^\sigma}(\cdot)$  and  $\Psi^{\Psi^\sigma}(\cdot)$ , respectively.*

**Proposition 5.3.** *Consider Assumption 5.2. Let also  $(x, y, z)$  be a solution of FBSDEs (36),  $(\mathcal{Y}, \Psi^\mathcal{Y})$  be a random field pair that belongs to  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^3))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}^3))$  and satisfies the BSPDE*

$$\begin{aligned}
d\mathcal{Y} = & \left\{ -\frac{1}{2} \sigma^2(t) \mathcal{Y}_{xx} - \frac{1}{2} z^2 \mathcal{Y}_{yy} - \frac{1}{2} h^2 \mathcal{Y}_{zz} - \sigma(t) z \mathcal{Y}_{xy} - \sigma(t) h \mathcal{Y}_{xz} - z h \mathcal{Y}_{yz} \right. \\
& + \sigma^2(t) \mathcal{Y} \mathcal{Y}_x + \sigma(t) z \mathcal{Y} \mathcal{Y}_y + \sigma(t) h \mathcal{Y} \mathcal{Y}_z + \sigma(t) m(t, x) \mathcal{Y}_x \\
& \left. - \left( d(t, x) z + c(t, x) y - m(t, x) z \right) \mathcal{Y}_y \right. \\
& - \left[ \left( A^\sigma(t) + \sigma^2(t) d_x(t, x(t)) - d(t, x(t)) \Psi^\sigma(t) \right. \right. \\
& \quad \left. \left. + \sigma(t) c(t, x(t)) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right) \frac{z(t)}{\sigma(t)} \right. \\
& + \frac{1}{2} \sigma^2(t) \Psi_{xx}^V(t, x(t)) - \left( \sigma(t) d(t, x(t)) + \Psi^\sigma(t) \right) \Psi_x^V(t, x(t)) \\
& - \left( A^\sigma(t) - d(t, x(t)) \Psi^\sigma(t) \right. \\
& \quad \left. + \sigma(t) c(t, x(t)) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right) \frac{\Psi^V(t, x(t))}{\sigma(t)} \\
& + A^{\Psi^V}(t) + \sigma(t) \Psi_x^{\Psi^V}(t, x(t)) - \left( d(t, x(t)) + \frac{\Psi^\sigma(t)}{\sigma(t)} \right) \Psi^{\Psi^V} \\
& \left. + \sigma(t) c_x(t, x(t)) y(t) + h \left( \frac{\Psi^\sigma(t)}{\sigma(t)} + d(t, x(t)) - m(t, x) \right) \right] \mathcal{Y}_z \\
& + e(t, x) \mathcal{Y} + \sigma(t) \mathcal{Y} \Psi^\mathcal{Y} - \sigma(t) \Psi_x^\mathcal{Y} - z \Psi_y^\mathcal{Y} - h \Psi_z^\mathcal{Y} + m(t, x) \Psi^\mathcal{Y} \\
& \left. + f(t, x) \right\} dt + \Psi^\mathcal{Y} dW(t) \quad \text{on} \quad [0, T) \times \mathbb{R}^3
\end{aligned} \tag{38}$$

for any  $h \in \mathbb{R}$ , and

$$\mathcal{Z} \triangleq \Psi^\mathcal{Y} + \sigma(t) \mathcal{Y}_x + z(t) \mathcal{Y}_y + h(t) \mathcal{Y}_z \tag{39}$$

evaluated at  $(t, x(t), y(t), z(t))$  on  $[0, T]$ . Then the process triplet  $(X, Y, Z)$  given by the point transformation (37) solves the FBSDEs (33).

*Proof.* Firstly we provide the semimartingale decomposition of the process  $z(\cdot)$  in (35). Given Assumption 5.2, considering that the pair  $(V, \Psi^V)$  belongs to the class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^3(\mathbb{R}))$ , and assuming that the random field  $\Psi^V$  has a semimartingale decomposition for  $(A^{\Psi^V}, \Psi^{\Psi^V}) \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^1(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$ , employ the product rule and the generalized IKW formula to process  $z(\cdot)$  of (35), in conjunction with (34), to get that

$$\begin{aligned} dz(t) = & \left\{ A^{\Psi^V}(t, x(t)) + \sigma(t) \left( \sigma(t) d(t, x(t)) + \Psi^\sigma(t) \right) V_{xx}(t, x(t)) \right. \\ & + \left[ A^\sigma(t) + \sigma^2(t) d_x(t, x(t)) + \sigma(t) c(t, x(t)) \right] V_x(t, x(t)) \\ & + \sigma(t) c_x(t, x(t)) V(t, x(t)) + \frac{1}{2} \sigma^2(t) \Psi_{xx}^V(t, x(t)) \\ & + \left( \sigma(t) d(t, x(t)) + \Psi^\sigma(t) \right) \Psi_x^V(t, x(t)) \\ & \left. + \sigma(t) d_x(t, x(t)) \Psi^V(t, x(t)) + \sigma(t) \Psi_x^{\Psi^V}(t, x(t)) \right\} dt \\ & + \left[ \Psi^{\Psi^V}(t, x(t)) + \sigma^2(t) V_{xx}(t, x(t)) \right. \\ & \left. + \Psi^\sigma(t) V_x(t, x(t)) + 2\sigma(t) \Psi_x^V(t, x(t)) \right] dW(t) \end{aligned}$$

for  $0 \leq t < T$ . Therefore, set

$$h(t) \triangleq \Psi^{\Psi^V}(t, x(t)) + \sigma^2(t) V_{xx}(t, x(t)) + \Psi^\sigma(t) V_x(t, x(t)) + 2\sigma(t) \Psi_x^V(t, x(t))$$

for every  $t \in [0, T]$  and recall the third equation of (35) to derive the relationships

$$\begin{aligned} V_x(t, x(t)) &= \frac{z(t) - \Psi^V(t, x(t))}{\sigma(t)}, \\ V_{xx}(t, x(t)) &= \frac{h(t) - \Psi^{\Psi^V}(t, x(t))}{\sigma^2(t)} + \frac{\Psi^\sigma(t) \left( \Psi^V(t, x(t)) - z(t) \right)}{\sigma^3(t)} \\ &\quad - \frac{2\Psi_x^V(t, x(t))}{\sigma(t)}, \end{aligned} \tag{40}$$

and deduce eventually for any  $t \in [0, T)$  the semimartingale decomposition

$$\begin{aligned}
dz(t) = & \left\{ \left[ A^\sigma(t) + \sigma^2(t)d_x(t, x(t)) - \Psi^\sigma(t)d(t, x(t)) \right. \right. \\
& \left. \left. + \sigma(t)c(t, x(t)) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right] \frac{z(t)}{\sigma(t)} \right. \\
& + \frac{1}{2} \sigma^2(t)\Psi_{xx}^V(t, x(t)) - \left( \sigma(t)d(t, x(t)) + \Psi^\sigma(t) \right) \Psi_x^V(t, x(t)) \\
& - \left[ A^\sigma(t) - \Psi^\sigma(t)d(t, x(t)) \right. \\
& \left. \left. + \sigma(t)c(t, x(t)) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right] \frac{\Psi^V(t, x(t))}{\sigma(t)} \right. \\
& + A^{\Psi^V}(t) + \sigma(t)\Psi_x^{\Psi^V}(t, x(t)) - \left( d(t, x(t)) + \frac{\Psi^\sigma(t)}{\sigma(t)} \right) \Psi^{\Psi^V} \\
& \left. + \sigma(t)c_x(t, x(t))y(t) + \left( \frac{\Psi^\sigma(t)}{\sigma(t)} + d(t, x(t)) \right) h(t) \right\} dt + h(t)dW(t).
\end{aligned} \tag{41}$$

We are ready now to argue that BSPDE (38) constitutes a sufficient condition for the point transformation (37) to produce solutions of the FBSDEs (33) directly out of solutions of the FBSDEs (36). Indeed, this follows directly from the generalized IKW formula applied to the second equation of (37), in combination with (36), (41), and (38).  $\square$

We shall focus next on establishing an equivalent characterization for the solutions of BSPDE (38), in order to determine the point transformation of (37). A byproduct of this analysis will be the Cole-Hopf transformation between the solutions  $U$  and  $V$  of the BSPDEs (31) and (34), respectively; cf. Remark 5.5.

**Proposition 5.4.** *Let Assumption 5.2 hold and set*

$$d(t, x) \triangleq m(t, x) \quad \text{on } [0, T] \times \mathbb{R}. \tag{42}$$

*Then the BSPDE of (38) admits solutions of the form*

$$\mathcal{Y}(t, x, y, z) = -\frac{z}{\sigma(t)y - r(t, x)} + \frac{\Psi^\sigma(t)}{\sigma^2(t)} + \frac{\sigma(t)r_x(t, x) - \Psi^\sigma(t)y + \Psi^r(t, x)}{\sigma(t)(\sigma(t)y - r(t, x))} \tag{43}$$

*on  $[0, T) \times \mathbb{R}^3$ , where*

$$e(t, x) \triangleq \sigma(t)m_x(t, x) \quad \text{and} \quad c(t, x) \triangleq -\int f(t, x)dx + \bar{c}(t) \quad \text{on } [0, T] \times \mathbb{R} \tag{44}$$

for some stochastic process  $\bar{c}(\cdot)$ , the random field  $r : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  satisfies the linear BSPDE

$$\begin{aligned} dr = & \left\{ -\frac{1}{2} \sigma^2(t) r_{xx} + \left( \Psi^\sigma(t) + \sigma(t) m(t, x) \right) r_x \right. \\ & + \left[ \sigma(t) c(t, x) - \Psi^\sigma(t) m(t, x) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} + A^\sigma(t) \right] \frac{r}{\sigma(t)} - \sigma(t) \Psi_x^r \\ & \left. + \left( \frac{\Psi^\sigma(t)}{\sigma(t)} + m(t, x) \right) \Psi^r \right\} dt + \Psi^r dW(t) \quad \text{on} \quad [0, T] \times \mathbb{R}, \end{aligned} \quad (45)$$

and the constraint

$$\begin{aligned} & \left[ -\frac{2(\Psi^\sigma(t))^2}{\sigma(t)} + \Psi^{\Psi^\sigma}(t) \right] r_x \\ & + \left[ 5 \frac{(\Psi^\sigma(t))^3}{\sigma^2(t)} - 3 \frac{\Psi^{\Psi^\sigma}(t) \Psi^\sigma(t)}{\sigma(t)} - \Psi^{\Psi^\sigma}(t) m(t, x) \right. \\ & \quad \left. + \frac{2(\Psi^\sigma(t))^2 m(t, x)}{\sigma(t)} - \frac{2A^\sigma(t) \Psi^\sigma(t)}{\sigma(t)} + A^{\Psi^\sigma}(t) \right] \frac{r}{\sigma(t)} \\ & - \frac{1}{2} \sigma^2(t) \Psi_{xx}^r + \left( 2\Psi^\sigma(t) + \sigma(t) m(t, x) \right) \Psi_x^r \\ & + \left[ 2 \frac{A^\sigma(t)}{\sigma(t)} - 5 \frac{(\Psi^\sigma(t))^2}{\sigma^2(t)} + \frac{\Psi^{\Psi^\sigma}(t)}{\sigma(t)} + c(t, x) - 2 \frac{\Psi^\sigma(t) m(t, x)}{\sigma(t)} \right] \Psi^r \\ & - \sigma(t) \Psi_x^{\Psi^r} + \left( 2 \frac{\Psi^\sigma(t)}{\sigma(t)} + m(t, x) \right) \Psi^{\Psi^r} + \frac{1}{2} \sigma^3(t) \Psi_{xx}^V \\ & - \sigma(t) \left( \sigma(t) m(t, x) + \Psi^\sigma(t) \right) \Psi_x^V \\ & - \left( A^\sigma(t) - \Psi^\sigma(t) m(t, x) + \sigma(t) c(t, x) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right) \Psi^V \\ & + \sigma(t) A^{\Psi^V} + \sigma^2(t) \Psi_x^{\Psi^V} - \left( \sigma(t) m(t, x) + \Psi^\sigma(t) \right) \Psi^{\Psi^V} - A^{\Psi^r} = 0 \end{aligned} \quad (46)$$

holds on  $[0, T] \times \mathbb{R}$ .

*Proof.* Since (38) holds for all  $h \in \mathbb{R}$ , we see that from the term quadratic in  $h$  we get  $\mathcal{Y}_{zz} = 0$ , which on the real line shows that

$$\mathcal{Y}(t, x, y, z) = \mathcal{P}(t, x, y) z + \mathcal{Q}(t, x, y) \quad \text{on} \quad [0, T] \times \mathbb{R}^3 \quad (47)$$

for appropriate random fields  $\mathcal{P}, \mathcal{Q} \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^2)))$ , that each obtains a semimartingale decomposition. Additionally, from the terms linear in  $h$  and (42) we find that

$$-\sigma(t) \mathcal{Y}_{xz} - z \mathcal{Y}_{yz} + \sigma(t) \mathcal{Y} \mathcal{Y}_z - \frac{\Psi^\sigma(t)}{\sigma(t)} \mathcal{Y}_z - \Psi_z^{\mathcal{Y}} = 0;$$

substituting (47), the left-hand-side becomes a linear polynomial with respect to  $z$  which implies that

$$\mathcal{P}(t, x, y) = -\frac{1}{\sigma(t)y - r(t, x)} \quad \text{and} \quad (48)$$

$$\mathcal{Q}(t, x, y) = \frac{\Psi^\sigma(t)}{\sigma^2(t)} + \frac{\sigma(t)r_x(t, x) - \Psi^\sigma(t)y + \Psi^r(t, x)}{\sigma(t)(\sigma(t)y - r(t, x))} \quad (49)$$

on  $[0, T] \times \mathbb{R}^2$ , in terms of a random field  $r : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  with a semimartingale decomposition to be determined.

As a consequence of this analysis, the BSPDE (38) takes the reduced form

$$\begin{aligned} d\mathcal{Y} = & \left\{ -\frac{1}{2} \sigma^2(t) \mathcal{Y}_{xx} - \frac{1}{2} z^2 \mathcal{Y}_{yy} - \sigma(t) z \mathcal{Y}_{xy} + \sigma^2(t) \mathcal{Y} \mathcal{Y}_x + \sigma(t) z \mathcal{Y} \mathcal{Y}_y \right. \\ & - \left[ \left( A^\sigma(t) + \sigma^2(t) m_x(t, x(t)) \right. \right. \\ & \quad \left. \left. - \Psi^\sigma(t) m(t, x(t)) + \sigma(t) c(t, x(t)) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right) \frac{z(t)}{\sigma(t)} \right. \\ & \quad \left. + \frac{1}{2} \sigma^2(t) \Psi_{xx}^V(t, x(t)) - \left( \sigma(t) m(t, x(t)) + \Psi^\sigma(t) \right) \Psi_x^V(t, x(t)) \right. \\ & \quad \left. - \left( A^\sigma(t) - \Psi^\sigma(t) m(t, x(t)) \right. \right. \\ & \quad \left. \left. + \sigma(t) c(t, x(t)) - \frac{(\Psi^\sigma(t))^2}{\sigma(t)} \right) \frac{\Psi^V(t, x(t))}{\sigma(t)} + A^{\Psi^V}(t) \right. \\ & \quad \left. + \sigma(t) \Psi_x^{\Psi^V}(t, x(t)) - \left( m(t, x(t)) + \frac{\Psi^\sigma(t)}{\sigma(t)} \right) \Psi^{\Psi^V} \right. \\ & \quad \left. \left. + \sigma(t) c_x(t, x(t)) y(t) \right] \mathcal{Y}_z \right. \\ & \quad \left. + \sigma(t) m(t, x) \mathcal{Y}_x - c(t, x) y \mathcal{Y}_y + e(t, x) \mathcal{Y} + \sigma(t) \mathcal{Y} \Psi^\mathcal{Y} - \sigma(t) \Psi_x^\mathcal{Y} \right. \\ & \quad \left. - z \Psi_y^\mathcal{Y} + m(t, x) \Psi^\mathcal{Y} + f(t, x) \right\} dt + \Psi^\mathcal{Y} dW(t) \quad \text{on } [0, T] \times \mathbb{R}^3, \end{aligned} \quad (50)$$

and from (47), (48), and (49) obtains solutions in the form of (43). Through substitution of (47) and comparing the finite-variation terms of the two sides, we arrive again at an equation between

two linear polynomials with respect to  $z$  which yields the relationships:

$$\begin{aligned}
A^{\mathcal{P}} = & -\frac{1}{2} \sigma^2(t) \mathcal{P}_{xx} - \sigma(t) \mathcal{Q}_{xy} + \sigma^2(t) \mathcal{Q} \mathcal{P}_x + \sigma^2(t) \mathcal{P} \mathcal{Q}_x + \sigma(t) \mathcal{Q} \mathcal{Q}_y \\
& + \sigma(t) m(t, x) \mathcal{P}_x - c(t, x) y \mathcal{P}_y + e(t, x) \mathcal{P} + \sigma(t) \mathcal{P} \Psi^{\mathcal{Q}} + \sigma(t) \mathcal{Q} \Psi^{\mathcal{P}} \\
& - \left[ A^{\sigma}(t) + \sigma^2(t) m_x(t, x) + \sigma(t) c(t, x) \right. \\
& \quad \left. - \Psi^{\sigma}(t) m(t, x) - \frac{(\Psi^{\sigma}(t))^2}{\sigma(t)} \right] \frac{\mathcal{P}}{\sigma(t)} \\
& - \sigma(t) \Psi_x^{\mathcal{P}} - \Psi_y^{\mathcal{Q}} + m(t, x) \Psi^{\mathcal{P}},
\end{aligned} \tag{51}$$

$$\begin{aligned}
A^{\mathcal{Q}} = & -\frac{1}{2} \sigma^2(t) \mathcal{Q}_{xx} + \sigma(t) \left( \sigma(t) \mathcal{Q} + m(t, x) \right) \mathcal{Q}_x - c(t, x) y \mathcal{Q}_y + e(t, x) \mathcal{Q} \\
& - \left[ \frac{1}{2} \sigma^2(t) \Psi_{xx}^V(t, x(t)) - \left( \sigma(t) m(t, x(t)) + \Psi^{\sigma}(t) \right) \Psi_x^V(t, x(t)) \right. \\
& \quad \left. - \left( A^{\sigma}(t) - \Psi^{\sigma}(t) m(t, x(t)) \right. \right. \\
& \quad \left. \left. + \sigma(t) c(t, x(t)) - \frac{(\Psi^{\sigma}(t))^2}{\sigma(t)} \right) \frac{\Psi^V(t, x(t))}{\sigma(t)} \right. \\
& \quad \left. + A^{\Psi^V}(t) + \sigma(t) \Psi_x^{\Psi^V}(t, x(t)) - \left( m(t, x(t)) + \frac{\Psi^{\sigma}(t)}{\sigma(t)} \right) \Psi^{\Psi^V} \right. \\
& \quad \left. + \sigma(t) c_x(t, x(t)) y \right] \mathcal{P} - \sigma(t) \Psi_x^{\mathcal{Q}} + \left( \sigma(t) \mathcal{Q} + m(t, x) \right) \Psi^{\mathcal{Q}} + f(t, x),
\end{aligned}$$

both on  $[0, T) \times \mathbb{R}^2$ . Employ Itô differentials to (48) and (49) to compute explicitly the semimartingale decompositions of  $\mathcal{P}$  and  $\mathcal{Q}$ , and make the substitutions in (51) to end up at two zero-polynomials in  $y$ , of first and second order respectively, which imply eventually the definitions of (44), the BSPDE (45), and the constraint (46).  $\square$

BSPDE (45) admits the apparent solution of  $r(\cdot, \cdot) = 0$ ; then, (43) simplifies to

$$\mathcal{Y}(t, x, y, z) = -\frac{z}{\sigma(t)y} \quad \text{on} \quad [0, T) \times \mathbb{R}^3, \tag{52}$$

as well as, solving (46) for  $A^{\Psi^V}$  and substituting back to the semimartingale decomposition of  $\Psi^V$ , we get the BSPDE constraint

$$\begin{aligned}
d\Psi^V = & \left\{ -\frac{1}{2} \sigma^2(t) \Psi_{xx}^V + \left[ \sigma(t) m(t, x) + \Psi^{\sigma}(t) \right] \Psi_x^V \right. \\
& + \left( \frac{A^{\sigma}(t)}{\sigma(t)} - \frac{\Psi^{\sigma}(t)}{\sigma(t)} m(t, x) + c(t, x) - \frac{(\Psi^{\sigma}(t))^2}{\sigma(t)^2} \right) \Psi^V \\
& \left. - \sigma(t) \Psi_x^{\Psi^V} + \left( \frac{\Psi^{\sigma}(t)}{\sigma(t)} + m(t, x) \right) \Psi^{\Psi^V} \right\} dt + \Psi^{\Psi^V} dW(t).
\end{aligned} \tag{53}$$



on  $[0, T] \times \mathbb{R}$ . Moreover, combining (32), (35), (37), and (52) we have that

$$\begin{aligned} U(t, x(t)) &= Y(t) = \mathcal{Y}(t, x(t), y(t), z(t)) = -\frac{z(t)}{\sigma(t)y(t)} \\ &= -\frac{V_x(t, x(t))}{V(t, x(t))} - \frac{\Psi^V(t, x(t))}{\sigma(t)V(t, x(t))}, \end{aligned} \quad (54)$$

which is always the case if

$$U = -\frac{\partial}{\partial x} \ln V - \frac{\Psi^V}{\sigma(t)V} \quad (55)$$

holds for every  $(t, x) \in [0, T] \times \mathbb{R}$ . In fact for  $t = T$ , putting together the terminal conditions of (31) and (34), we get that the random field  $q$  must satisfy the condition

$$p(x) = -\frac{\partial}{\partial x} \ln q(x) - \frac{\Psi^V(T, x)}{\sigma(T)q(x)}, \quad x \in \mathbb{R}. \quad (56)$$

**Remark 5.5.** *The relationship of (55) is the generalized Cole-Hopf transformation that provides the pattern to construct solutions  $(U, \Psi^U)$  of the stochastic Burgers BSPDE (31) from solutions  $(V, \Psi^V)$  of the stochastic linear heat equation (34), subject to the additional linear constraint of (53) and the terminal condition of (56); a verification result is provided below (cf. Theorem 5.6). Apparently in a deterministic setting (55) reduces to the standard Cole-Hopf transformation of Section 3 and (53) becomes a tautology.*

*One can immediately see that considering either deterministic or random coefficients the point transformations  $\mathcal{Y}$  of (11) and (37) are both of the same form; cf. Remark 3.3 and (52). However, the application of the probabilistic characterization of the Cole-Hopf transformation, deployed in the previous section, led to Proposition 5.3 and eventually concluded to the additional constraint for the random field  $\Psi^V$  of (53), which vanishes in the deterministic case.*

**Theorem 5.6.** *Consider Assumption 5.2, definitions (42) and (44), and a random field pair  $(V, \Psi^V) \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^3(\mathbb{R}))$  that satisfies the heat-type BSPDE (34) with  $V(\cdot, \cdot) > 0$ , subject to the constraints (53) for the pair  $(\Psi^V, \Psi^{\Psi^V}) \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$  and (56) for the random field  $q$ . Then the random field pair  $(U, \Psi^U)$  of class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$  defined through (55) is a solution of the Burgers-type BSPDE (31).*

**Remark 5.7.** *It is important at this point to make a comment on the constraints as expressed in equations (53) and (56). For a given final condition  $q(x)$ , the backward heat equation (34) admits a unique solution pair  $(V, \Psi^V)$ . Then, the uniqueness of the semimartingale decomposition for the random field  $\Psi^V$  provides us with a uniquely determined  $\Psi^{\Psi^V}$ . Therefore, a fruitful way of handling these constraints is to substitute in them the pair  $(\Psi^V, \Psi^{\Psi^V})$ , as defined by the solution of (34), and then find the general class of coefficients of the original problem so that the constraints are satisfied. This approach provides us with large families of exactly solvable backward Burgers equations with random coefficients; see Subsection 5.3.*

**Remark 5.8.** It is worth mentioning that, regarding (42), (44), and solution pairs  $(V, \Psi^V)$  and  $(r, \Psi^r)$  of the BSPDEs (34) and (45), respectively, the random field  $\mathcal{Y}$  of (37) defined through (43) provides the most general form of solution to (38), subject to the constraint (46). In view of (54) and (55) we obtain here the relationship

$$U = -\frac{V_x}{V-r} - \frac{\Psi^V}{\sigma(t)V-r} + \frac{\Psi^\sigma(t)}{\sigma^2(t)} + \frac{\sigma(t)r_x - \Psi^\sigma(t)V + \Psi^r}{\sigma(t)(\sigma(t)V-r)}$$

for every  $(t, x) \in [0, T) \times \mathbb{R}$ , which establishes the most general formulation of the Cole-Hopf transformation between the BSPDEs of (31) and (34).

*Proof.* OF THEOREM 5.6: For  $0 \leq t < T$  and  $x \in \mathbb{R}$ , Itô's rule, (55), (34), and (53) yield

$$\begin{aligned} dU = & \left\{ \sigma(t) \frac{\Psi_{xx}^V}{2V} - \left( \sigma(t)m(t, x) + \Psi^\sigma(t) \right) \frac{\Psi_x^V}{\sigma(t)V} + \Psi^\sigma(t)m(t, x) \frac{\Psi^V}{\sigma^2(t)V} \right. \\ & + \frac{\Psi_x^V}{V} - m(t, x) \frac{\Psi^V}{\sigma(t)V} - \sigma(t) \frac{\Psi^V V_{xx}}{2V^2} + m(t, x) \frac{\Psi^V V_x}{V^2} \\ & + m(t, x) \frac{(\Psi^V)^2}{\sigma(t)V^2} - \frac{\Psi^V \Psi_x^V}{V^2} - \frac{(\Psi^V)^3}{\sigma(t)V^3} - \Psi^\sigma(t) \frac{(\Psi^V)^2}{\sigma^2(t)V^2} + \frac{\Psi^V \Psi^{\Psi^V}}{\sigma(t)V^2} \\ & + \sigma^2(t) \frac{V_{xxx}}{2V} - \sigma^2(t) \frac{V_{xx} V_x}{2V^2} - \sigma(t)m_x(t, x) \frac{V_x}{V} - \sigma(t)m(t, x) \frac{V_{xx}}{V} \\ & + \sigma(t)m(t, x) \frac{(V_x)^2}{V^2} - m_x(t, x) \frac{\Psi^V}{V} - m(t, x) \frac{\Psi_x^V}{V} + m(t, x) \frac{\Psi^V V_x}{V^2} \\ & \left. + \sigma(t) \frac{\Psi_{xx}^V}{V} - \sigma(t) \frac{\Psi_x^V V_x}{V^2} - c_x(t, x) + \frac{\Psi_x^V \Psi^V}{V^2} - \frac{(\Psi^V)^2 V_x}{V^3} \right\} dt \\ & + \left\{ -\frac{\Psi^V}{\sigma(t)V} + \frac{(\Psi^V)^2}{\sigma(t)V^2} + \Psi^\sigma(t) \frac{\Psi^V}{\sigma^2(t)V} - \frac{\Psi_x^V}{V} + \frac{\Psi^V V_x}{V^2} \right\} dW(t). \end{aligned} \quad (57)$$

Setting

$$\Psi^U = -\frac{\Psi^V}{\sigma(t)V} + \frac{(\Psi^V)^2}{\sigma(t)V^2} + \Psi^\sigma(t) \frac{\Psi^V}{\sigma^2(t)V} - \frac{\Psi_x^V}{V} + \frac{\Psi^V V_x}{V^2},$$

in conjunction with (55), it is straightforward to obtain that

$$\begin{aligned} \Psi_x^U = & -\frac{\Psi_x^V}{\sigma(t)V} + \frac{\Psi^V V_x}{\sigma(t)V^2} + 2 \frac{\Psi^V \Psi_x^V}{\sigma(t)V^2} - 2 \frac{(\Psi^V)^2 V_x}{\sigma(t)V^3} + \Psi^\sigma(t) \frac{\Psi_x^V}{\sigma^2(t)V} \\ & - \Psi^\sigma(t) \frac{\Psi^V V_x}{\sigma^2(t)V^2} - \frac{\Psi_{xx}^V}{V} + 2 \frac{\Psi_x^V V_x}{V^2} + \frac{\Psi^V V_{xx}}{V^2} - 2 \frac{\Psi^V (V_x)^2}{V^3}, \\ U_x = & -\frac{\Psi_x^V}{\sigma(t)V} + \frac{\Psi^V V_x}{\sigma(t)V^2} - \frac{V_{xx}}{V} + \frac{(V_x)^2}{V^2}, \end{aligned}$$

$$\begin{aligned}
UU_x &= \frac{\Psi^V \Psi_x^V}{\sigma^2(t)V^2} - \frac{(\Psi^V)^2 V_x}{\sigma^2(t)V^3} + \frac{\Psi^V V_{xx}}{\sigma(t)V^2} + \frac{\Psi_x^V V_x}{\sigma(t)V^2} - 2 \frac{\Psi^V (V_x)^2}{\sigma(t)V^3} \\
&\quad + \frac{V_{xx} V_x}{V^2} - \frac{(V_x)^3}{V^3}, \\
U\Psi^U &= \frac{\Psi^V \Psi \Psi^V}{\sigma^2(t)V^2} - \frac{(\Psi^V)^3}{\sigma^2(t)V^3} - \Psi^\sigma(t) \frac{(\Psi^V)^2}{\sigma^3(t)V^2} + \frac{\Psi^V \Psi_x^V}{\sigma(t)V^2} + \frac{\Psi \Psi^V V_x}{\sigma(t)V^2} \\
&\quad - 2 \frac{(\Psi^V)^2 V_x}{\sigma(t)V^3} + \Psi^\sigma(t) \frac{\Psi^V V_x}{\sigma^2(t)V^2} + \frac{\Psi_x^V V_x}{V^2} - \frac{(V_x)^2 \Psi^V}{V^3}, \\
U_{xx} &= -\frac{\Psi_{xx}^V}{\sigma(t)V} + \frac{\Psi^V V_{xx}}{\sigma(t)V^2} - \frac{V_{xxx}}{V} + 3 \frac{V_{xx} V_x}{V^2} - 2 \frac{(V_x)^3}{V^3} \\
&\quad + 2 \frac{\Psi_x^V V_x}{\sigma(t)V^2} - 2 \frac{(V_x)^2 \Psi^V}{\sigma(t)V^3}.
\end{aligned}$$

Therefore, recalling (42) and (44), direct substitution of the above into (31) leads to (57) and completes the proof.  $\square$

Theorem 5.6 necessitates the investigation of existence, uniqueness, and positivity of solutions for the linear BSPDE (34) subject to the constraints (53), which is another linear BSPDE for their martingale part, and (56). Ma & Yong ([34], [36]) studied linear BSPDEs of parabolic type and provided regularity conditions on their coefficients that ensured existence, uniqueness, and comparison results between solutions. However, their results are invalid under the presence of a BSPDE constraint.

**Corollary 5.9.** *Let Assumption 5.2 hold and  $V$  belongs to the class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R})))$  be a positive random field that satisfies the BSPDE*

$$\begin{aligned}
dV &= \left\{ -\frac{1}{2} \sigma^2(t) V_{xx} + \sigma(t) m(t, x) V_x + c(t, x) V \right\} dt, \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\
V(T, x) &= q(x), \quad x \in \mathbb{R},
\end{aligned}$$

where the random field  $c$  is given by (44) and  $q$  is defined as in (21). Then the random field

$$U = -\frac{\partial}{\partial x} \ln V \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))),$$

defined through the Cole-Hopf transformation, solves the Burgers-type BSPDE

$$dU = \left[ -\frac{1}{2} \sigma^2(t) U_{xx} + \sigma^2(t) U U_x + \sigma(t) m(t, x) U_x + \sigma(t) m_x(t, x) U + f(t, x) \right] dt, \quad 0 \leq t < T, \quad x \in \mathbb{R},$$

$$U(T, x) = p(x), \quad x \in \mathbb{R}.$$

*Proof.* The statement of the theorem follows as a direct application of Theorem 5.6 to the random field pair  $(V, 0)$ ; in particular, the constraint (53) holds trivially and (56) yields the definition of  $q$  as in (21).  $\square$

### 5.3 Examples of completely solvable backward random Burgers equations

In what follows, we present examples of random field pairs  $(V, \Psi^V)$  that satisfy the hypotheses of Theorem 5.6 for *appropriate* coefficients.

**Example 5.10.** We postulate positive solutions for (34) of the form

$$V(t, x) \triangleq e^{f_1(x)W(t)} \quad (58)$$

on  $[0, T] \times \mathbb{R}$ , in terms of a given function  $f_1 : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C(\mathbb{R}^4)$ . From Itô's formula we have

$$dV = f_1(x) V dW(t) + \frac{1}{2} f_1^2(x) V dt, \quad (59)$$

which shows that

$$\Psi^V = f_1(x) V. \quad (60)$$

Then, the terminal condition of (34) indicates that the random field  $q$  is determined by

$$q(x) = e^{f_1(x)W(T)}, \quad x \in \mathbb{R}, \quad (61)$$

thus the random field  $p$  of (31) must be chosen according to (56). Using (58) and (60), we derive that

$$\begin{aligned} V_x &= f_1'(x) V W(t), \\ V_{xx} &= f_1''(x) V W(t) + (f_1'(x))^2 V W^2(t), \\ \Psi_x^V &= f_1'(x) V + f_1(x) f_1'(x) V W(t), \\ \Psi_{xx}^V &= f_1''(x) V + 2(f_1'(x))^2 V W(t) + f_1(x) f_1''(x) V W(t) \\ &\quad + f_1(x) (f_1'(x))^2 V W^2(t), \end{aligned}$$

and substituting them into (34), in comparison with (59), we get

$$\begin{aligned} \frac{1}{2} f_1^2(x) &= -\frac{1}{2} \sigma^2(t) \left[ f_1''(x) + (f_1'(x))^2 W(t) \right] W(t) + \sigma(t) m(t, x) f_1'(x) W(t) \\ &\quad + m(t, x) f_1(x) - \sigma(t) f_1'(x) [1 + f_1(x) W(t)] + c(t, x). \end{aligned} \quad (62)$$

Furthermore, Itô's formula and (60) give

$$d\Psi^V = f_1^2(x)VdW(t) + \frac{1}{2}f_1^3(x)Vdt, \quad (63)$$

which implies that

$$\begin{aligned} \Psi^{\Psi^V} &= f_1^2(x)V, \\ \Psi_x^{\Psi^V} &= 2f_1(x)f_1'(x)V + f_1^2(x)f_1'(x)VW(t); \end{aligned}$$

then, substitute these equations into (53) and compare it with (63) to get

$$\begin{aligned} \frac{1}{2}f_1^3(x) &= -\frac{1}{2}\sigma^2(t)\left[f_1''(x) + 2(f_1'(x))^2W(t) \right. \\ &\quad \left. + f_1(x)f_1''(x)W(t) + f_1(x)(f_1'(x))^2W^2(t)\right] \\ &\quad + (\sigma(t)m(t, x) + \Psi^\sigma(t))f_1'(x)[1 + f_1(x)W(t)] \\ &\quad + \left(\frac{A^\sigma(t)}{\sigma(t)} - \frac{\Psi^\sigma(t)}{\sigma(t)}m(t, x) + c(t, x) - \frac{(\Psi^\sigma(t))^2}{\sigma^2(t)}\right)f_1(x) \\ &\quad - \sigma(t)f_1(x)f_1'(x)[2 + f_1(x)W(t)] + \left(\frac{\Psi^\sigma(t)}{\sigma(t)} + m(t, x)\right)f_1^2(x). \end{aligned} \quad (64)$$

Obviously the random field triplet  $(V, \Psi^V, \Psi^{\Psi^V})$  belongs to the class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R}))) \times C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$ , according to Theorem 5.6. Furthermore, equations (62) and (64) form a  $2 \times 2$  linear system which can be easily solved for the random fields  $m$  and  $c$ . A selection of these coefficients together with (61) constitute the pair  $(V, \Psi^V)$  of (58) and (60) as a positive solution of (34) subject to the constraint (53).

**Example 5.11.** Alternatively, for a given positive function  $f_2 : \mathbb{R} \rightarrow \mathbb{R}^+$  of class  $C(\mathbb{R}^4)$ , we consider that (34) admits a solution of the form

$$V(t, x) \triangleq f_2(x)e^{W(t)} \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R}))). \quad (65)$$

Now, Itô's rule yields

$$dV = VdW(t) + \frac{1}{2}Vdt \quad \text{and} \quad \Psi^V = V, \quad (66)$$

the terminal condition of (34) imposes that the random field  $q$  must be given by

$$q(x) = f_2(x)e^{W(T)}, \quad x \in \mathbb{R}, \quad (67)$$

and the constraint (56) designates the random field  $p$  of (31) that should be considered. Therefore, differentiating (65) we obtain

$$V_x = f_2'(x)e^{W(t)} \quad \text{and} \quad V_{xx} = f_2''(x)e^{W(t)},$$

and through substitution back to (34) and (53), a comparison with (66) reveals that

$$\begin{aligned}
\frac{1}{2}f_2(x) &= -\frac{1}{2}\sigma^2(t)f_2''(x) + \sigma(t)m(t,x)f_2'(x) + m(t,x)f_2(x) \\
&\quad - \sigma(t)f_2'(x) + c(t,x)f_2(x), \\
\frac{1}{2}f_2(x) &= -\frac{1}{2}\sigma^2(t)f_2''(x) + \left(\sigma(t)m(t,x) + \Psi^\sigma(t)\right)f_2'(x) \\
&\quad + \left(\frac{A^\sigma(t)}{\sigma(t)} - \frac{\Psi^\sigma(t)}{\sigma(t)}m(t,x) + c(t,x) - \frac{(\Psi^\sigma(t))^2}{\sigma^2(t)}\right)f_2(x) \\
&\quad - \sigma(t)f_2'(x) + \left(\frac{\Psi^\sigma(t)}{\sigma(t)} + m(t,x)\right)f_2(x)
\end{aligned}$$

should hold respectively. Similarly to (i), the solution of this  $2 \times 2$  linear system determines the proper random fields  $m$  and  $c$  for which the pair  $(V, \Psi^V)$  of (65) and (66) is a positive solution of (34) subject to the constraint (53).

## 6 Stochastic Feynman-Kac Formulae

In this section we establish *stochastic* Feynman-Kac results for the solutions of the Burgers FSPDE (20) and BSPDE (31); cf. Propositions 6.1 and 6.2 below, respectively. Both results rely on the Cole-Hopf transformation, in its simple or generalized version (cf. (55)) respectively, which reduces the initial SPDEs to linear ones.

**Proposition 6.1.** *Let the assumptions of Theorem 4.1 hold, let the processes  $\sigma, \bar{c}$  and the random fields  $b, m$  be constants, let  $f(\cdot, \cdot) = 0$ , as well as  $p \in C^2(\mathbb{R})$ . If, in addition, there exist constants  $K > 0$  and  $0 < \alpha < 1/2T$  such that*

$$\max_{0 \leq t \leq T} |G(t, x)| + \max_{0 \leq t \leq T} |G_x(t, x)| \leq Ke^{\alpha x^2} \quad \forall x \in \mathbb{R},$$

*then the solution random field  $U$  of the corresponding Burgers FSPDE (20) admits for every  $(t, x) \in [0, T] \times \mathbb{R}$  and a.s. the stochastic representation*

$$U(t, x) = -\frac{G_x(t, x + H(t))}{G(t, x + H(t))},$$

where

$$G(t, x) = e^{-\bar{c}t} E^x \left[ q(x(t)) \right] \quad \text{and} \quad G_x(t, x) = e^{-\bar{c}t} E^x \left[ q_x(x(t)) \right];$$

here, the notation  $E^x$  stands for the expectation that corresponds to the process

$$\begin{aligned}
dx(t) &= k dt + \sigma dW(t), \quad 0 < t \leq T, \\
x(0) &= x.
\end{aligned}$$

*Proof.* According to hypothesis, differentiation of (25) implies that the random field  $G_x$  solves the same FSPDE with  $G$  but with initial condition given by  $G_x(0, x) = q_x(x)$  for  $x \in \mathbb{R}$ . Then the conclusion of the proposition is obtained by the Cole-Hopf transformation, (23), and standard Feynman-Kac results (e.g. [28] Corollary 4.4.5) applied to the FSPDEs whose solutions are the random fields  $G$  and  $G_x$ .  $\square$

**Proposition 6.2.** *Consider the FBSDEs of (36) on the interval  $[t, T]$  with  $x(t) = x$  for some  $0 \leq t \leq T$ , and assume that the hypotheses of Theorem 5.6 hold with  $m(\cdot, \cdot) = d(\cdot, \cdot) = \Psi^\sigma(\cdot) = 0$ . Then the solution  $(U, \Psi^U)$  of the Burgers BSPDE (31), corresponding to (55), obtains for all  $(t, x) \in [0, T] \times \mathbb{R}$  and a.s. the representation*

$$U(t, x) = -\frac{z^x(t)}{\sigma(t)y^x(t)},$$

where

$$\begin{aligned} z^x(t) = E^x \left[ -e^{-\int_t^T \left( \frac{A^\sigma(s)}{\sigma(s)} + c(s, x(s)) \right) ds} \sigma(T)p(x(T))q(x(T)) \right. \\ \left. + \int_t^T e^{-\int_t^s \left( \frac{A^\sigma(\theta)}{\sigma(\theta)} + c(\theta, x(\theta)) \right) d\theta} \sigma(s)f(s, x(s))y^x(s)ds \middle| \mathcal{F}(t) \right], \end{aligned}$$

$$y^x(t) = E^x \left[ q(x(T))e^{-\int_t^T c(s, x(s))ds} \middle| \mathcal{F}(t) \right];$$

here, we denote by  $E^x$  the expectation that corresponds to the forward process of the aforementioned FBSDE system.

*Proof.* Thanks to hypothesis and the constraint (53), the BSDEs of (36) and (41) simplify to

$$\begin{aligned} dy(s) &= c(s, x(s))y(s)ds + z(s)dW(s), \quad t \leq s < T, \quad \text{and} \\ dz(s) &= \left[ \left( \frac{A^\sigma(s)}{\sigma(s)} + c(s, x(s)) \right) z(s) - \sigma(s)f(s, x(s))y(s) \right] ds + h(s)dW(s), \end{aligned}$$

for  $t \leq s < T$ , respectively. Denoting their solutions by  $y^x(\cdot)$  and  $z^x(\cdot)$ , observe that these BSDEs are linear with drift terms independent of their auxiliary processes  $z(\cdot)$  and  $h(\cdot)$ , respectively. Additionally, set  $t = T$  in (54) and invoke the terminal conditions of (31) and (36) to verify that the terminal condition of the last BSDE above is given by

$$z(T) = -\sigma(T)p(x(T))q(x(T)). \quad (68)$$

Therefore, the assertion of the proposition follows as an application of Corollary 6.2 by Ma & Yong (1997), in combination with (54), the above BSDEs, and their terminal conditions in (36) and (68).  $\square$

## 7 Applications

The following applications illustrate the use of Theorem 5.6 to construct solutions of backward stochastic Burgers equations with random coefficients, which are less commonly found in the literature. These equations turn out to be associated with problems on controllability or mathematical finance.

### 7.1 Controllability of a backward random Burgers equation

Consider the following control system

$$dU = \left[ -\frac{1}{2} \sigma^2(t, x) U_{xx} + a(t, x) U U_x + g(t, x) U U_1 + b(t, x) U_x + e(t, x) U + s(t, x) U_2 + m(t, x) U_1 + f(t, x) \right] dt + U_1 dW(t), \quad 0 \leq t < T, \quad x \in \mathbb{R}, \quad (69)$$

where  $(U_1, U_2)$  is a pair of control processes to be determined and  $p_0(x)$  is the initial state of the system. The question we wish to answer is the following. Setting a desired final condition  $p(x)$  for the control system (69), for which initial state  $p_0(x)$  does there exist a pair of control processes  $(U_1, U_2)$  such that this system can be driven in time  $T$  to the desired state  $p(x)$ ?

The linearization results for the backward Burgers equation may prove useful to answering this question. Assume that the two control procedures are connected by  $U_2 = \frac{\partial}{\partial x} U_1$ . Then the control system is equivalent to the BSPDE (5) with  $U_1 = \Psi^U$ . Therefore, having obtained a solution  $(U, \Psi^U)$  to problem (5) through the analysis of Section 5, we immediately obtain the control procedures needed to drive the system to the desired final state as  $U_1 = \Psi^U$  and  $U_2 = \Psi_x^U$ , and we characterize the initial state we need to start from by  $p_0(x) = U(0, x)$ .

### 7.2 Pricing a contingent claim

We consider a financial market on a finite time-horizon  $[0, T]$  consisting of a money market and a stock, whose prices  $S_0(\cdot)$  and  $S(\cdot)$  evolve according to the SDEs

$$\begin{aligned} dS_0(t) &= r(t) S_0(t) dt, \quad 0 < t \leq T, \quad S_0(0) = 1, \quad \text{and} \\ dS(t) &= S(t) [\mu(t) dt + \sigma(t) dW(t)], \quad 0 < t \leq T, \quad S(0) = s \geq 0, \end{aligned} \quad (70)$$

respectively. The interest rate  $r(\cdot) \geq 0$ , the instantaneous rate of return  $\mu(\cdot)$ , and the volatility  $\sigma(\cdot)$  are taken to be bounded,  $\mathbb{F}$ -progressively measurable random processes with  $\sigma(\cdot) \neq 0$ .

In this market, we consider an economic agent with an initial endowment  $Y(0)$ , who at any time  $t \in [0, T]$  invests a proportion  $\pi(t)$  of his wealth  $Y(t)$  in the stock, saves the remaining amount  $[1 - \pi(t)]Y(t)$  in the money market, consumes with a predetermined rate  $C(t) \geq 0$ , and is obliged to pay taxes with rate  $L(t) \geq 0$ . Here the investor is not allowed to sell short stocks or to borrow money at the bond interest rate  $r(\cdot)$ ; namely, the  $\pi(\cdot)$  above is restricted to take values only in  $[0, 1]$ . Furthermore, the tax regulation of the market mandates payments proportional to the size of the wealth  $Y(\cdot)$  with respect to the interest rate  $r(\cdot)$ , but also applies in favor of investing to the



stock by offering tax alleviation analogous to the size of the investment  $\pi(\cdot)$  and to the undertaken risk captured by the risky asset's volatility  $\sigma(\cdot)$ ; in particular, the tax rate is given by

$$L(t) = -\pi(t)\sigma^2(t)Y^2(t) + r(t)Y(t).$$

Therefore, in accordance with the market dynamics of (70), the *wealth process*  $Y(\cdot) \equiv Y^{\pi, C}(\cdot)$ , corresponding to the portfolio-consumption pair  $(\pi, C)$ , is the solution of the following SDE

$$\begin{aligned} dY(t) &= (1 - \pi(t))r(t)Y(t)dt + \pi(t)Y(t)\left[\mu(t)dt + \sigma(t)dW(t)\right] \\ &\quad - C(t)dt - L(t)dt \\ &= \left\{ \pi(t)(\mu(t) - r(t))Y(t) + \pi(t)\sigma^2(t)Y^2(t) - C(t) \right\} dt \\ &\quad + \pi(t)\sigma(t)Y(t)dW(t). \end{aligned} \tag{71}$$

Of course, all the preceding decisions taken by either the agent or the government should depend on the information available up to  $t$  and not anticipate the future, thus the *portfolio strategy*  $\pi : [0, T] \times \Omega \rightarrow \mathbb{R}$  and the *consumption strategy*  $C : [0, T] \times \Omega \rightarrow [0, \infty)$  are assumed to be  $\mathbb{F}$ -progressively measurable processes, and in addition verify the technical integrability condition  $\int_0^T (C(t) + \pi^2(t))dt < \infty$ , almost surely.

Let us now broach in this market the issue of pricing a contingent claim which depends on the volatility of the stock; that is, an  $\mathcal{F}(T)$ -measurable random variable  $\xi \geq 0$ , satisfying proper integrability conditions, of the form  $\xi = p(X(T))$ , where  $X(\cdot)$  is the forward process of (33) and  $p : \mathbb{R} \times \Omega \rightarrow [0, \infty)$  is an  $\mathcal{F}_T$ -measurable random field. This random amount  $\xi$  represents a liability for its seller that has to be covered with the smallest amount  $Y(0)$  of initial funds at time  $t = 0$  and the right trading strategy  $\pi(\cdot)$  during the interval  $[0, T]$ , so that the corresponding wealth process  $Y(t)$  is positive for all  $t \in [0, T]$  and at the end of the time-horizon  $Y(T) = p(X(T))$  holds without risk. In other words, to hedge and determine the fair price  $Y(0)$  of the contingent claim at time  $t = 0$  it suffices to find the solution pair  $(Y, \pi)$  of the BSDE (71) subject to the previous terminal condition such that  $Y(\cdot) > 0$ .

Setting

$$Z(t) \triangleq \pi(t)\sigma(t)Y(t), \quad 0 \leq t < T, \tag{72}$$

and eliminating the portfolio  $\pi(\cdot)$  in the drift term of (71), we get the BSDE

$$\begin{aligned} dY(t) &= \left[ \sigma(t)Z(t)Y(t) + m(t)Z(t) - C(t) \right] dt + Z(t)dW(t), \quad 0 \leq t < T, \\ Y(T) &= p(X(T)), \end{aligned}$$

where

$$m(\cdot) \triangleq \frac{\mu(\cdot) - r(\cdot)}{\sigma(\cdot)}$$

is the well known relative market of risk process. Due to Proposition 5.1, its solution is given by the pair  $(Y, Z)$  of (32), where  $(U, \Psi^U)$  is of class  $C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$  and

solves the BSPDE

$$\begin{aligned} dU &= \left[ -\frac{1}{2} \sigma^2(t) U_{xx} + \sigma^2(t) U U_x + \sigma(t) U \Psi^U + m(t) \sigma(t) U_x - \sigma(t) \Psi_x^U \right. \\ &\quad \left. + m(t) \Psi^U - C(t) \right] dt + \Psi^U dW(t), \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\ U(T, x) &= p(x), \quad x \in \mathbb{R}. \end{aligned} \quad (73)$$

In turn, Theorem 5.6 provides the solution of this BSPDE via the generalized Cole-Hopf transformation (55), in terms of a triplet  $(V, \Psi^V, \Psi^{\Psi^V}) \in C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^4(\mathbb{R}))) \times C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}))) \times \mathbb{L}_{\mathbb{F}}^2(0, T; C^2(\mathbb{R}))$  that solves the linear BSPDE

$$\begin{aligned} dV &= \left\{ -\frac{1}{2} \sigma^2(t) V_{xx} + \sigma(t) m(t) V_x - \sigma(t) \Psi_x^V + m(t) \Psi^V + C(t) x V \right\} dt \\ &\quad + \Psi^V dW(t), \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\ V(T, x) &= q(x), \quad x \in \mathbb{R} \end{aligned}$$

for  $\bar{c}(\cdot) = 0$ , subject to  $V(\cdot, \cdot) > 0$ ,

$$\begin{aligned} d\Psi^V &= \left\{ -\frac{1}{2} \sigma^2(t) \Psi_{xx}^V + \left[ \sigma(t) m(t) + \Psi^\sigma(t) \right] \Psi_x^V \right. \\ &\quad \left. + \left( \frac{A^\sigma(t)}{\sigma(t)} - m(t) \frac{\Psi^\sigma(t)}{\sigma(t)} + C(t) x - \frac{(\Psi^\sigma(t))^2}{\sigma^2(t)} \right) \Psi^V - \sigma(t) \Psi_x^{\Psi^V} \right. \\ &\quad \left. + \left( \frac{\Psi^\sigma(t)}{\sigma(t)} + m(t) \right) \Psi^{\Psi^V} \right\} dt + \Psi^{\Psi^V} dW(t) \quad \text{on} \quad [0, T) \times \mathbb{R}, \end{aligned}$$

and (56). Then, the hedging portfolio process  $\pi(\cdot)$  follows immediately from (72).

Finally, we shall illustrate simple and explicit solutions of the above system in the case of constant model coefficients  $r, \mu, \sigma$ , so as  $m$ , and zero consumption, i.e.,  $C(\cdot) \equiv 0$ . In light of Examples 5.10 and 5.11, we postulate strictly positive solutions of the form

$$V(t, x) \triangleq e^{\alpha W(t)} \quad \text{or} \quad V(t, x) \triangleq \beta e^{W(t)}$$

for given constants  $\alpha, \beta > 0$ , and take  $\sigma < 0$  so that (55) establishes positive solutions for the BSPDE (73), respectively. Then, in order for the corresponding  $2 \times 2$  systems in these examples to hold, it suffices to select the coefficients of the model such that

$$m(\cdot) = \frac{1}{2} \alpha \quad \text{or} \quad m(\cdot) = \frac{1}{2},$$

and from (56)

$$p(\cdot) = -\frac{\alpha}{\sigma} > 0 \quad \text{or} \quad p(\cdot) = -\frac{1}{\sigma} > 0,$$

respectively.

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